University of Pisa

Doctoral Thesis

Dehn Surgery on the Minimally Twisted Five-Chain Link

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September 2011

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Introduction

Surgery was introduced by Dehn in [**Deh**] to construct homology spheres. The technique was later shown by Lickorish [**Lic**] to be a means of describing all closed orientable 3-manifolds, and with the emergence of Thurston's geometrization program (see [**Th2**]) surgery theory has become one of the central themes of low-dimensional topology.

One of the objects of considerable interest in the literature has been the set of non-hyperbolic Dehn fillings of a given hyperbolic manifold. The size of such a set is known to be at most 10 [**LM**]. Moreover, additional restrictions exist on the set. For example, no manifold has two S^3 fillings. This is the content of the celebrated Gordon-Luecke theorem which says that every knot is determined by its complement up to orientation-preserving homeomorphism [**GL2**].

Many questions about the set of non-hyperbolic Dehn fillings of a hyperbolic manifold remain unanswered, for example see [**KL**]. This thesis is about such sets. In particular, we provide a full classification of the exceptional surgeries on an explicit hyperbolic link known to be relevant to many important problems in the literature.

Terminology

We begin with some general terminology, and we introduce the notion of an "exceptional filling instruction" used in the statements of the main results in this thesis.

Fix an orientable compact 3-manifold X with ∂X consisting of tori:

- A *slope* on a boundary component τ of X is the isotopy class of a non-trivial unoriented loop on τ;
- A *filling instruction* α for X is a set consisting of either a slope or the empty set for each component of ∂X;

- The *filling* $X(\alpha)$ given by an instruction α is the manifold obtained by attaching one solid torus to ∂X for each (non-empty) slope in α, with the meridian of the solid torus attached to the slope.

We recall that if M is a hyperbolic non-compact finite-volume 3-manifold then $M = \mathrm{int}(X)$ with ∂X consisting of tori, and that $\mathrm{int}(X(\alpha))$ is hyperbolic for all but finitely many α's consisting of one slope and $\varnothing$'s.

- If the interior of X is hyperbolic but the interior of $X(\alpha)$ is not, we say that α is an *exceptional filling instruction* for X and that $X(\alpha)$ is an *exceptional filling* of X;
- We say that an exceptional filling instruction α on a hyperbolic X is *isolated* if $X(\beta)$ is hyperbolic for all β properly contained in α; for such an α we call $X(\alpha)$ an *isolated exceptional filling* of X.

A *surgery* on a link L in a 3-manifold M corresponds to a filling of the exterior of L, that is, to a filling of $M \backslash N(L)$ where $N(L)$ is an open regular neighbourhood of L. By *surgery instruction* for L we mean a filling instruction on the exterior of L.

Exceptional pairs and the Gordon program

This section is based on the notes from a course delivered by Cameron Gordon at the ICPT Trieste, in the Summer of 2009, [**G1**].

The JSJ decomposition and Geometrization theorems tell us that every non-hyperbolic 3-manifold not homeomorphic to the 3-ball either contains an essential sphere, disc, torus or annulus, or is a closed small Seifert space.

The closed small Seifert spaces are precisely those manifolds with Heegaard genus 0, Heegaard genus 1 or that fibre over the sphere with exactly 3 exceptional fibres. See Section 1.2 for full definitions and statements of the JSJ decomposition and Geometrization theorems and further details. We now assign names to each class of non-hyperbolic manifolds:

- The class of Heegaard genus 0 manifolds (*i.e.* $\{S^3\}$) is denoted by S^H;
- The class of all reducible 3-manifolds is denoted by S;
- The class of manifolds with Heegaard genus 1 (*i.e.* lens space) is denoted by T^H;
- The class of manifolds containing an essential torus is denoted by T;
- The class of boundary reducible manifolds is denoted by D;
- The class of manifolds containing an essential annulus is denoted by A;

- The class of Seifert spaces fibering over the sphere with exactly three exceptional fibres is denoted by Z.

We will say that a manifold in a class $\mathcal{C}$ is of *type* $\mathcal{C}$. We remark that the above classes are not mutually exclusive, for example $(D^2 \times S^1)\#(D^2 \times S^1)$ is of type S and of type D.

Let M be a fixed orientable cusped hyperbolic 3-manifold and let τ be a fixed toric component of the boundary of its compactification. Given two classes of non-hyperbolic manifolds $\mathcal{C}_1$, $\mathcal{C}_2 \in \{S^H, S, T^H, T, D, A, Z\}$ and slopes α_1, α_2 on τ with $M(\alpha_i)$ of type $\mathcal{C}_i$, we will say that $(M, \tau; \alpha_1, \alpha_2)$ is an *exceptional pair* of type $(\mathcal{C}_1, \mathcal{C}_2)$. In the case when M has a single boundary component, we will denote an exceptional pair by $(M; \alpha_1, \alpha_2)$.

Definition Let α_1 and α_2 be two slopes on a torus. The *minimal geometric intersection number* of α_1 and α_2, or equivalently the *distance* between α_1 and α_2, is defined to be $\min\{|a_1 \pitchfork a_2|\}$ where $a \in \alpha_1$ and $a_2 \in \alpha_2$, and it is denoted by $\Delta(\alpha_1, \alpha_2)$.

Let X be a 3-manifold, let τ be a toroidal boundary component of ∂X, and let μ, λ be two non-trivial oriented curves on τ with $\Delta(\mu, \lambda) = 1$. For a slope α on τ we have $[\alpha] = \pm(m[\mu] + l[\lambda]) \in H_1(\tau)$ for some coprime m and l. Identifying α with $\frac{m}{l}$ we then get a bijection between the set of slopes on τ and $\mathbb{Q} \cup \{\infty\}$. When the (μ, λ) pair on the boundary component of a 3-manifold is clear we often write $\frac{m}{l}$ to denote the slope $\pm(m[\mu] + l[\lambda])$.

There is a preferred choice of (μ, λ) for surgery on the boundary of a regular neighbourhood of a knot K in S^3; let $N(K) \cong K \times \mathrm{int}(D^2)$ be an open regular neighbourhood of K and let M_K denote $S^3 \backslash N(K)$; we set λ to be an oriented curve in ∂M_K parallel to K with $[\lambda] = 0 \in H_1(M_K)$, and $\mu = \{*\} \times \partial D^2$, oriented so that μ, λ is a positive basis of $H_1(\tau)$ with τ oriented as ∂M_K. The curve μ is called a *meridian* of K and λ a *longitude* of K.

It turns out that the minimal geometric intersection number is easily computed (see [**Sti**] for details):

Proposition 0.0.1. *Let τ be a torus and (μ, λ) be a basis of $H_1(\tau)$. For slopes $\alpha_1 = \frac{p}{q}$ and $\alpha_2 = \frac{r}{s}$ one has $\Delta(\alpha_1, \alpha_2) = |ps - rq|$.*

It is a consequence of [**LM**] that 8 is a universal upper bound for $\Delta(\alpha_1, \alpha_2)$ for each exceptional pair $(M, \tau; \alpha_1, \alpha_2)$.

Definition Define $\Delta_0(\mathcal{C}_1, \mathcal{C}_2)$ to be the greatest value of $\Delta(\alpha_1, \alpha_2)$ among exceptional pairs $(M, \tau; \alpha_1, \alpha_2)$ of type $(\mathcal{C}_1, \mathcal{C}_2)$.

Using this language, the Gordon-Luecke theorem can be formulated by saying that $\Delta_0(S^H, S^H) = 0$, the Cabling conjecture by saying that $\Delta_0(S, S^H) = -\infty$ [**Sch**], the Berge conjecture [**Ber**] by saying that the Berge knots exactly describe all exceptional pairs of type (S^H, T^H), and the theorem of [**GL1**] by saying that the knots realizing $\Delta(\alpha_1, \alpha_2) = \Delta_0(S^H, T)$ are precisely the Eudave-Muñoz knots.

We will now outline a systematic program for enumerating all exceptional pairs, called the Gordon program, see [**GL1**], [**KL**] and [**G1**] for discussion.

The Gordon program.

Problem 1: For each pair of classes $\mathcal{C}_1, \mathcal{C}_2 \in \{S^H, S, T^H, T, D, A, Z\}$, determine $\Delta_0(\mathcal{C}_1, \mathcal{C}_2)$.

Problem 2: Determine all $(M, \tau; \alpha_1, \alpha_2)$ of type $(\mathcal{C}_1, \mathcal{C}_2)$ with $\Delta(\alpha_1, \alpha_2) = \Delta_0(\mathcal{C}_1, \mathcal{C}_2)$.

Problem 3: Determine all $(M, \tau; \alpha_1, \alpha_2)$ of type $(\mathcal{C}_1, \mathcal{C}_2)$ with $\Delta(\alpha_1, \alpha_2) = \delta$ for every $\delta \le \Delta_0(\mathcal{C}_1, \mathcal{C}_2)$.

For this thesis, in connection with the Gordon program, we restrict our attention to fillings of manifolds with one boundary component so we do not look at the exceptional slopes of type D or A. An overview of what is known for Problem 1 for the five classes we shall consider is shown in Table 0.0.1, where (c) refers to the entry being conjectured, and (b) refers to the entry being the best known lower bound; in all other cases the entries represent proved values of $\Delta_0(\mathcal{C}_1, \mathcal{C}_2)$.

Δ_0	S	T	S^H	T^H	Z
S	1	3	$-\infty$ (c)	1	2 (b)
T		8	2	3 (b)	7 (b)
S^H			0	1	1 (b)
T^H				1	2 (b)
Z					6 (b)

TABLE 0.0.1. Maximal distances between exceptional surgeries.

References for the $\Delta_0(\mathcal{C}_1, \mathcal{C}_2)$'s of Table 0.0.1 are given in [**GL1**] and [**G1**]. In addition to the above remarks about $\Delta_0(S, S^H)$, $\Delta_0(S^H, T)$, and $\Delta_0(S^H, T^H)$, it should also be noted that $\Delta_0(T^H, T)$ is known to be at most 4 [**Lee**], and that the best lower bounds on $\Delta_0(Z, \mathcal{C})$ come from [**MP**].

Thesis problem and results

The main body of Chapters 1-4 is original, but it also includes standard results that are presented with references throughout. We use this section to state the main thesis problem and indicate some of the original results we have established.

We now introduce the main thesis problem. Denote the chain link of Figure 0.0.1 by 5CL. It is well known that 5CL is hyperbolic (see for example [**DT**]).

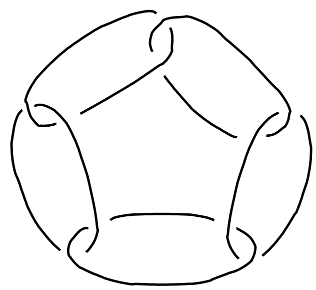


Figure 0.0.1. The minimally twisted 5-chain link.

Problem 1: Determine all the exceptional surgery instructions for 5CL and all the corresponding exceptional surgeries.

A full description of all exceptional surgery instructions on a link with five components is in general ambitious. However, there are two important points to note about the minimally twisted 5-chain link; firstly, part of the analysis is already known (see [**MP**], and Chapter 3 for details) and, secondly, the link exterior possesses a large symmetry group (see Theorem 1.0.7). These facts make the program of enumerating all exceptional surgery instructions achievable.

The main results of this thesis are Theorems 1.1.1 and 1.1.3. The former gives a complete description of all isolated exceptional surgery instructions for 5CL, and the latter gives a description of all exceptional surgery instructions for 5CL and an algorithm to write down the corresponding surgeries. In addition to this, Corollary 1.2.4 explicitly describes all the exceptional surgeries on 5CL. The statements of these results are long and therefore we only highlight here some of the consequences that they have.

Theorem 0.0.2. *Every exceptional surgery on* 5CL *is either homeomorphic to a surgery on the chain links* 3CL *or* M4CL *in Figure 0.0.2 or is one of the following manifolds:*

- $\big(D,(2,1),(2,-1)\big)\bigcup_{\begin{pmatrix}1&4\\1&3\end{pmatrix}}\big(D,(2,-3),(3,2)\big)$,
- $\big(D,(2,1),(2,-1)\big)\bigcup_{\begin{pmatrix}1&5\\1&4\end{pmatrix}}\big(D,(2,-3),(3,2)\big)$,
- $\big(A,(2,-1)\big)/_{\begin{pmatrix}1&2\\1&1\end{pmatrix}}$.

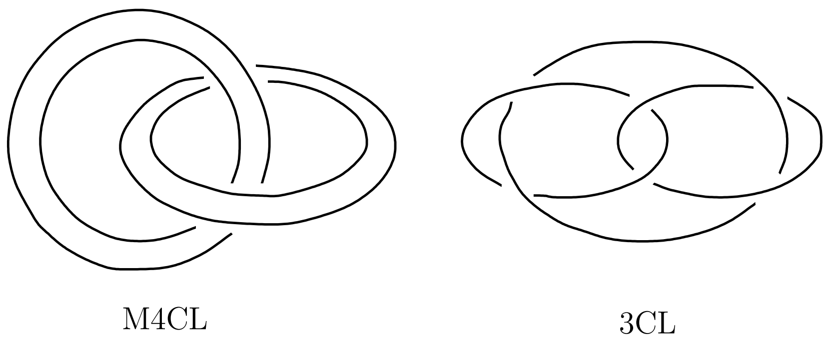


FIGURE 0.0.2. The chain links M4CL and 3CL.

THEOREM 0.0.3. *Every small Seifert manifold appears as an exceptional surgery on* 5CL.

Of course, when a hyperbolic surgery instruction on 5CL has less than 5 slopes then the corresponding surgery on 5CL is a hyperbolic manifold with boundary. Thus a full answer to Problem 1 generates many examples of hyperbolic manifolds.

Problem 2: Use the examples produced from the answer to Problem 1 to look at Problem 2 from the Gordon program.

This raises the following:

Question: Why the minimally twisted five chain link?

It turns out that most closed hyperbolic 3-manifolds from the Hodgson-Weeks census [**HW**] are surgeries on 5CL, see [**DT**]. Moreover, it turns out that 5CL relates to many problems from the Gordon program. In particular, all manifolds with pairs of slopes realising $\Delta_0(S^H,T)$ and $\Delta_0(S,T)$ are surgeries on 5CL, see [**GL1**] and [**Kan**] respectively. In addition to this, "many" of the hyperbolic manifolds realising $\Delta_0(S^H,T^H)$ are surgeries on 5CL, see [**Bak**].

In the course of our solution of Problem 1 we will see that many exceptional surgeries on 5CL are also exceptional surgeries on 3CL, that were classified in [**MP**], see Theorem 1.1.3.

Theorem 0.0.4. *There are exactly* 6 *hyperbolic surgeries on* 5CL *that are not surgeries on* 3CL *and possess more than* 5 *exceptional slopes.*

In Chapters 3 and 4 we begin the program of enumerating all exceptional $(5\mathrm{CL}(f), \tau; \alpha_1, \alpha_2)$ of type $(\mathcal{C}_1, \mathcal{C}_2)$ for f a filling instruction on 5CL and $\mathcal{C}_i \in \{S^H, S, T^H, T, Z\}$. In particular, we begin an explicit description of the exceptional slopes on $5\mathrm{CL}(f)$ and the corresponding exceptional surgeries when $5\mathrm{CL}(f)$ has 1 boundary component and $\Delta(\alpha_1, \alpha_2) = \Delta_0(\mathcal{C}_1, \mathcal{C}_2)$. We highlight here:

Theorem 0.0.5. *No exceptional pair* $(5\mathrm{CL}(f); \alpha_1, \alpha_2)$ *of type* (T^H, T) *with* $5\mathrm{CL}(f)$ *a one-cusped hyperbolic manifold has* $\Delta(\alpha_1, \alpha_2) = 4$.

Theorem 0.0.6. *Every exceptional pair* $(5\mathrm{CL}(f); \alpha_1, \alpha_2)$ *of type* (S, T) *with* $\Delta(\alpha_1, \alpha_2) = \Delta_0(S, T)$ *and* $5\mathrm{CL}(f)$ *a one-cusped hyperbolic manifold has* $e(5\mathrm{CL}(f)) = 5$. *In all but one case, these* $5\mathrm{CL}(f)$ *admit exactly* 1 *filling of type* T^H *and* 2 *fillings of type* Z.

In Chapter 4, we show that the problem of describing *all* $(5\mathrm{CL}(f), \tau; \alpha_1, \alpha_2)$ of type $(\mathcal{C}_1, \mathcal{C}_2)$ with $\Delta(\alpha_1, \alpha_2) = \Delta_0(\mathcal{C}_1, \mathcal{C}_2)$ and the corresponding exceptional fillings of $5\mathrm{CL}(f)$ is tractable. Particular questions of this type are asked in Sections 4.2-4.3. In addition to this we highlight the introduction of a class of toroidal manifold in Section 3.3 which we feel is interesting owing to the rarity with which manifolds in this class appear as exceptional surgeries of 5CL. We set out a research program about this class in Section 4.4.

An important point is the experimental nature of our results. In particular, the proofs that we present for our main results would have been quite unachievable without computer assistance. Our results are highly dependant on the computer program SnapPea [**We1**] which uses the Newton-Raphson method of approximation to find solutions to the hyperbolicity equations, see [**We2**]. Thus parts of the proofs are necessarily "black boxed". In addition to SnapPea, the programs SnapPy [**CDW**] and Recogniser [**Ma1**] were also used.

Thesis structure and presentation

The presentation of this thesis tries to be as self-contained as possible but assumes basic notions from topology and geometry. Detailed accounts of all assumed material can be found in [**FM**] and [**BP**].

Chapters 1 and 2 contain the solution to Problem 1. The classification of exceptional surgeries on 5CL is expressed in terms of exceptional fillings of the exterior M_5 of 5CL. Section 1.1 contains the statements of the main results of the thesis: the complete description of all isolated exceptional filling instructions for M_5 (Theorem 1.1.1), the complete description of all exceptional filling instructions for M_5 and of the corresponding exceptional fillings (Theorem 1.1.3 and Proposition 1.2.3). All exceptional fillings are described in Corollary 1.2.4, and the sets of exceptional slopes for the fillings of M_5 are described in Corollaries 1.3.1 and 1.3.2.

Chapter 2 contains complete proofs of all of the results stated in Chapter 1. The proofs of the statements found in Section 1.n are found in Section 2.n.

Chapters 3 and 4 deal with Problem 2. In Chapter 3, surgery instructions for 5CL "factoring through" the exterior M_3 of 3CL are investigated by directly considering filling instructions and fillings of M_3. In Propositions 3.1.1 and 3.1.3, we obtain a description of many of the isolated exceptional surgeries on 3CL. In Section 3.2 we deal with the surgeries on 3CL of type T^H. All surgeries on 3CL of type S^H are described in Proposition 3.2.1, all surgeries on 3CL with 3 fillings of type T^H, and all surgeries on 3CL that are hyperbolic knot exteriors in S^3 with 2 fillings of type T^H are described in Corollary 3.2.2. The Eudave-Muñoz knots appearing as surgeries on 3CL are described in Proposition 3.2.3. In Section 3.2 we enumerate all the remaining exceptional pairs at maximal distance coming from 3CL. In particular, for $M_3(f)$ a one-cusped hyperbolic manifold, Proposition 3.2.4 describes all $(M_3(f);\alpha_1,\alpha_2)$'s of type $(S,\mathcal{C})$, Proposition 3.2.5 describes all $(M_3(f);\alpha_1,\alpha_2)$'s of type $(T^H,\mathcal{C})$, and Proposition 3.2.8 describes all $(M_3(f);\alpha_1,\alpha_2)$'s of type $(Z,\mathcal{C})$. The Chapter ends in Section 3.3 with the definition of a class of toroidal manifolds that we denote by $\mathcal{I}$. Questions about the class $\mathcal{I}$ are later considered in Chapter 4.

Chapter 4 deals with the exceptional pairs in $(\mathcal{C}_1,\mathcal{C}_2)$ at maximal distance not coming from M_3. In Section 4.1 we set out a complete, but lengthy, description of the types of the exceptional 5CL$(f)(\alpha)$'s such that f does not factor through M_3 and 5CL(f) is a one-cusped hyperbolic manifold; Corollaries 4.1.2 and 4.1.3, and Proposition 4.1.4 give conditions for the class of each filling of $M_5(f)$ when f does not factor through M_3. In Section 4.2 we use the results from Section 4.1 to look at the program of describing all $(M_5(f);\alpha,\beta)$ of type $(\mathcal{C}_1,\mathcal{C}_2)$ with $\Delta(\alpha_1,\alpha_2)=\Delta_0(\mathcal{C}_1,\mathcal{C}_2)$ and $M_5(f)$ a one-cusped hyperbolic manifold. The main result of Section 4.2 is Theorem 4.2.2 where we give a complete description of all exceptional $(M_5(f);\alpha_1,\alpha_2)$'s of type (S,T) with $\Delta_0(\alpha_1,\alpha_2)=\Delta_0(S,T)$, and we give a complete description of the sets of the corresponding exceptional fillings. In Section 4.3 we examine the description of Berge knots in [**Bak**] using the results from Chapter 1. In particular we provide a full description of the sets of exceptional slopes on the "sporadic" Berge knots and the corresponding surgery types of these slopes in Proposition 4.3.2 and Corollary 4.3.3.

In Section 4.4 we give a full description of all one-cusped hyperbolic manifolds of the form $M_5(f)$ admitting a filling in $\mathcal{I}$ and the sets of exceptional slopes and the corresponding exceptional fillings of these manifolds. We end the thesis by outlining a research program centered on the class $\mathcal{I}$.

Acknowledgements

I am privileged to be able to warmly thank and acknowledge many people for their help and advise on this thesis.

In particular I can strongly thank my supervisors Prof. Petronio and Dr. Martelli, who, over the course of my time in Italy have mentored me in mathematics, in the writing and presentation of mathematics, and even in the writing of basic English! It's been an absolute pleasure for me, and I thank you deeply for your patience and all the time that you have given me.

Outside of Pisa, I have written e-mails to Dr. Weeks, Prof. Dunfield and Prof. Baker when I have had (often silly) questions during my studies and I can happily thank them for their always helpful and patient responses. I would also like to thank Prof. Gordon for the course that he delivered in Trieste in the Summer of 2008; it greatly influenced the direction of this thesis.

I must also mention Prof. Broglia and Prof. Acquistapace who gave me an immense amount of help in navigating some important (but impossible) Italian bureaucracy.

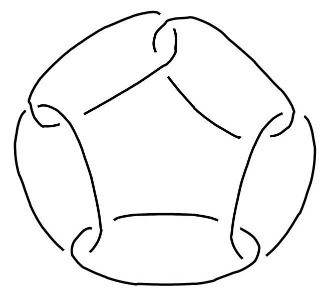
This brings me to the final sentences that I've waited three years to write, namely, to the personal acknowledgements of the good people of Santa Ciara and my troupe from around the world who kept me alive and fighting during the crazy

times in 2008. It's been more than 3 years since February 15th 2008 and I'm very alive, and I'm very well! We did it together. Your care, support, concern and love will never be forgotten and as an extremely small token I dedicate this thesis to you. You know who you are, and I offer you my most absolute humble and sincere thanks. Thank you.

CHAPTER 1

Classification

In this chapter we state our classification results for the exceptional surgeries on the minimally twisted 5-chain link. We recall some basics from the Introduction.



We denote the minimally twisted 5-chain link (pictured) by 5CL and the exterior of 5CL by M_5, so that Dehn surgeries on 5CL coincide with Dehn fillings of M_5. We fix on each boundary component T of M_5 the usual (meridian, longitude) homology basis (μ, λ), and we recall that a slope on T corresponds to $\frac{a}{b} \in \mathbb{Q} \cup \{\infty\}$ if its homology class is $\pm(a\mu + b\lambda)$. We number the boundary components of M_5 as $T_0, \ldots, T_4$, so that T_i and T_{i+1} correspond to linked components of 5CL (indices $\mod 5$).

Denote the set of all possible filling instructions on M_5 by $\mathcal{S}$. Using the meridian-longitude homology basis on each component of ∂M_5 and the numbering of these components, we identify $\mathcal{S}$ to $(\mathbb{Q} \cup \{\infty, \varnothing\})^5$. We denote the equivalence relation on $\mathcal{S}$ induced by the action of the symmetry group of M_5 by $\sim$.

THEOREM 1.0.7. *The equivalence relation $\sim$ on $\mathcal{S}$ is generated by the following maps:*

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) \longmapsto (\alpha_4, \alpha_0, \alpha_1, \alpha_2, \alpha_3) \tag{1.1}$$

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) \longmapsto (\alpha_4, \alpha_3, \alpha_2, \alpha_1, \alpha_0) \tag{1.2}$$

$$\left(\tfrac{a}{b}, \tfrac{c}{d}, \tfrac{e}{f}, \tfrac{g}{h}, \tfrac{i}{j}\right) \longmapsto \left(\tfrac{f}{e}, \tfrac{j-i}{j}, \tfrac{a}{a-b}, \tfrac{d-c}{d}, \tfrac{h}{g}\right) \tag{1.3}$$

$$\left(\tfrac{a}{b}, \tfrac{c}{d}, \tfrac{e}{f}, \tfrac{g}{h}, \tfrac{i}{j}\right) \longmapsto \left(\tfrac{b}{b-a}, \tfrac{i-j}{i}, \tfrac{e-f}{e}, \tfrac{d}{d-c}, \tfrac{g}{h}\right) \tag{1.4}$$

$$\left(\tfrac{a}{b}, \tfrac{c}{d}, \tfrac{e}{f}, \tfrac{g}{h}, \tfrac{i}{j}\right) \longmapsto \left(\tfrac{i}{i-j}, \tfrac{b-a}{b}, \tfrac{f}{e}, \tfrac{d}{c}, \tfrac{h-g}{h}\right) \tag{1.5}$$

$$\left(\tfrac{a}{b}, \tfrac{c}{d}, \tfrac{e}{f}, \tfrac{g}{h}, \tfrac{i}{j}\right) \longmapsto \left(\tfrac{j}{j-i}, \tfrac{e}{f}, \tfrac{b}{b-a}, \tfrac{c-d}{c}, \tfrac{g-h}{g}\right) \tag{1.6}$$

$$\left(\tfrac{a}{b},\tfrac{c}{d},\tfrac{e}{f},\tfrac{g}{h},\tfrac{i}{j}\right) \longmapsto \left(\tfrac{a}{a-b},\tfrac{e}{e-f},\tfrac{i}{i-j},\tfrac{c}{c-d},\tfrac{g}{g-h}\right) \tag{1.7}$$

$$\left(\tfrac{a}{b},\tfrac{c}{d},\tfrac{e}{f},\tfrac{g}{h},\tfrac{i}{j}\right) \longmapsto \left(\tfrac{h}{g},\tfrac{j}{i},\tfrac{f-e}{f},\tfrac{c}{c-d},\tfrac{b-a}{b}\right) \tag{1.8}$$

$$\left(\tfrac{a}{b},\tfrac{c}{d},\tfrac{e}{f},\tfrac{g}{h},\tfrac{i}{j}\right) \longmapsto \left(\tfrac{h}{h-g},\tfrac{a}{b},\tfrac{f}{f-e},\tfrac{c-d}{c},\tfrac{i-j}{i}\right) \tag{1.9}$$

$$\left(\tfrac{a}{b},\tfrac{c}{d},\tfrac{e}{f},\tfrac{g}{h},\tfrac{i}{j}\right) \longmapsto \left(\tfrac{g}{g-h},\tfrac{f-e}{f},\tfrac{b}{a},\tfrac{d}{c},\tfrac{j-i}{j}\right) \tag{1.10}$$

$$\left(\tfrac{a}{b},\tfrac{c}{d},\tfrac{e}{f},\tfrac{g}{h},\tfrac{i}{j}\right) \longmapsto \left(\tfrac{g-h}{g},\tfrac{f}{f-e},\tfrac{i}{j},\tfrac{d}{d-c},\tfrac{a-b}{a}\right) \tag{1.11}$$

$$\left(\tfrac{a}{b},\tfrac{c}{d},\tfrac{e}{f},\tfrac{g}{h},\tfrac{i}{j}\right) \longmapsto \left(\tfrac{h-g}{h},\tfrac{b}{a},\tfrac{j}{i},\tfrac{d-c}{d},\tfrac{e}{e-f}\right) \tag{1.12}$$

$$\left(\tfrac{a}{b},\tfrac{c}{d},\tfrac{e}{f},\tfrac{g}{h},\tfrac{i}{j}\right) \longmapsto \left(\tfrac{a-b}{a},\tfrac{e-f}{e},\tfrac{h}{h-g},\tfrac{c}{d},\tfrac{j}{j-i}\right). \tag{1.13}$$

Remark When a symbol appearing in the argument of one of these maps is $\varnothing$, there is a corresponding symbol $\varnothing$ in the image of the map.

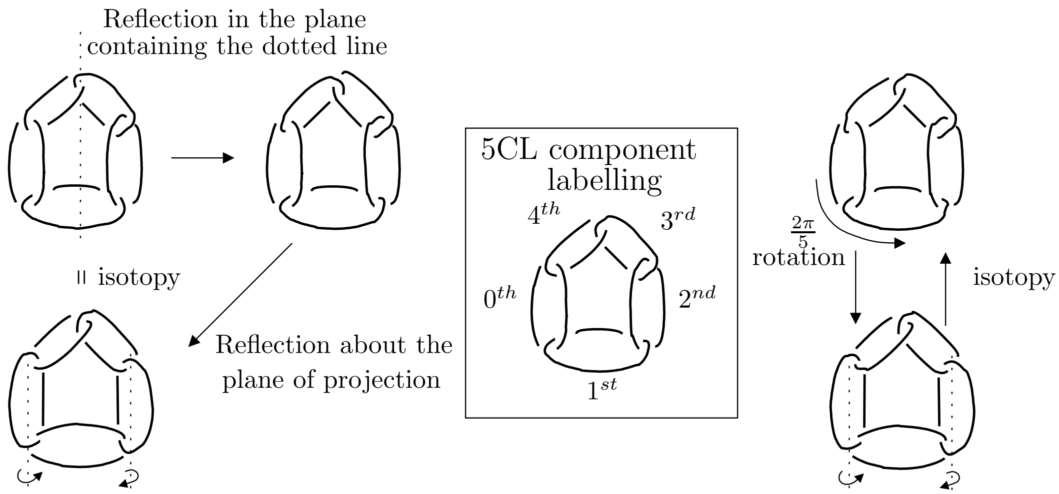
Remark The action of (1.1) and (1.2) on $\mathcal{S}$ corresponds to that of the dihedral group D_5, namely the symmetry of the link 5CL, which is of course a subgroup of the whole symmetry group of M_5. The generators of D_5 acting on 5CL are shown in Figure 1.0.1.

Since the action of the maps (1.1) and (1.2) is very easily understandable while that of (1.3)-(1.13) is more involved, we introduce the symbol $[\alpha]$ for the equivalence class of $\alpha \in \mathcal{S}$ under (1.1) and (1.2) only, and the symbol $[\![\alpha]\!]$ for the equivalence class of α under (1.1)-(1.13).

Remark If $\alpha \sim \beta$ then clearly $M_5(\alpha) \cong M_5(\beta)$, so in Theorem 1.1.1, our main classification result, we will only list the equivalence classes of isolated exceptional filling instructions $[\![\alpha]\!]$. For α an isolated exceptional filling instruction containing some $\varnothing$, we complement Theorem 1.1.1 with a description of the partition of $[\![\alpha]\!]$ into equivalence classes generated by (1.1)-(1.2) in Proposition 1.1.2.

1.1. Main results

To describe the isolated exceptional fillings of M_5 we introduce the links of Figure 1.1.1; the 3-chain link 3CL and the minimally twisted 4-chain link M4CL. We introduce another 4-chain link later in the thesis denoted 4CL. We denote the exterior of 3CL by M_3 and the exterior of M4CL by F. For each boundary component of both M_3 and F we use the intrinsic homology basis and number the

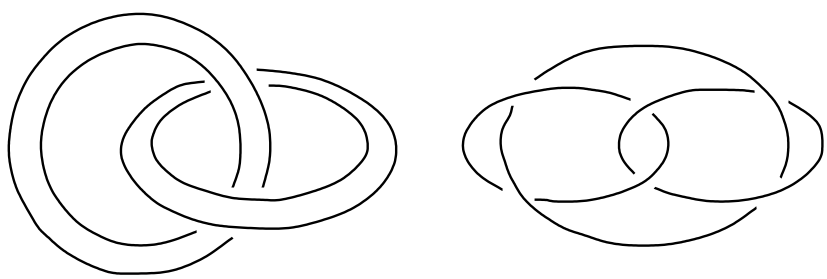


An order-2 symmetry of 5CL switching component 0 with component 2 and component 4 with component 3

Order-5 cyclic action on 5CL

Figure 1.0.1. Generators of the symmetry group of 5CL inducing a D_5 action on M_5.

boundary components of F as $T_0, \ldots, T_3$, so that T_i and T_{i+1} correspond to linked components of M4CL (indices $\mod 4$).



M4CL

3CL

Figure 1.1.1. Two links appearing in the classification of exceptional fillings of M_5; M4CL and 3CL. Their exteriors are denoted by F and M_3 respectively.

Remark It is easy to see that the symmetry group of 3CL is S_3 and so for a filling instruction α on M_3 we can write $M_3(\alpha)$ unambiguously. We identify filling instructions on F with elements of $(\mathbb{Q} \cup \{\infty, \varnothing\})^4$ and denote the equivalence relation on $(\mathbb{Q} \cup \{\infty, \varnothing\})^4$ induced from the symmetry group of M4CL by $\sim$.

It is easy to see that the symmetry of MT4C is the Dihedral group D_4. For a filling instruction α on F, by $[\alpha]$ we mean the equivalence class of α up to $\sim$. The hyperbolic structure on M_3 is well know (see [**MP**] for example), and F is non-hyperbolic (there are essential annuli evident in Figure 1.1.1)

Remark For a filling instruction α on M_5 we will often simplify notation by omitting empty slopes but leaving the subscripts on non-empty slopes. For example, $((-1)_1, (-1)_3)$ corresponds to the filling instruction $(\varnothing, \mu_1 - \lambda_1, \varnothing, \mu_3 - \lambda_3, \varnothing)$ with (μ_i, λ_i) the (meridian, longitude) basis of the homology of the i^{th} cusp. Note that for any $\frac{p}{q} \in \mathbb{Q} \cup \{\infty\}$ and $i \neq j$ one has $((\frac{p}{q})_i) \sim ((\frac{p}{q})_j)$ for slopes on M_5 or F, so the fillings $M_5(\frac{p}{q})$ and $F(\frac{p}{q})$ are defined without ambiguity. Our convention will be that a filling instruction $(\frac{p}{q}, \frac{r}{s}, \frac{u}{v}, \frac{x}{y})$ on M_5 with four non-empty slopes and no subscripts represents $(\frac{p}{q}, \frac{r}{s}, \frac{u}{v}, \frac{x}{y}, \varnothing)$.

Lastly we set

$$\begin{aligned} m_1 &= \big(D, (2,1), (2,-1)\big) \bigcup_{\begin{pmatrix} 1 & 4 \\ 1 & 3 \end{pmatrix}} \big(D, (2,-3), (3,2)\big) \\ m_2 &= \big(A, (2,-1)\big) \big/ {\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}} \\ m_3 &= \big(D, (2,1), (2,-1)\big) \bigcup_{\begin{pmatrix} 1 & 5 \\ 1 & 4 \end{pmatrix}} \big(D, (2,-3), (3,2)\big). \end{aligned}$$

THEOREM 1.1.1. *The equivalence classes under $\sim$ of all isolated exceptional filling instructions α of M_5 and the corresponding filled manifolds are given in Tables 1.1.1, 1.1.2, and 1.1.3, where $M_5(\alpha)$ is written as a filling of M_3 or of F, or as m_1, m_2, m_3.*

PROPOSITION 1.1.2. *The equivalence classes of filling instructions under (1.1)-(1.13) listed in Table 1.1.1 consist of the equivalence classes under (1.1)-(1.2) found in Tables 1.1.4-1.1.5.*

Number of filled cusps	Equivalence class under (1.1)-(1.13) of isolated exceptional filling instructions	Filled manifold
1	$\llbracket(1)\rrbracket$	F
2	$\llbracket((-1)_0, (-1)_1)\rrbracket$	$F(2)$
3	$\llbracket((-1)_0, (-2)_1, (-1)_2)\rrbracket$, $\llbracket((-2)_0, (-1)_1, (-2)_2)\rrbracket$	$M_3(0)$
	$\llbracket((-1)_0, 3_2, (-2)_4)\rrbracket$	$M_3(3)$
4	$\llbracket(-1,-2,-2,-1)\rrbracket$, $\llbracket(-1,-3,-1,-2)\rrbracket$, $\llbracket(-2,-2,-1,-3)\rrbracket$	$M_3(-1,-1)$
	$\llbracket(-1,-2,\frac{1}{2},\frac{5}{2})\rrbracket$, $\llbracket(-2,-1,-\frac{1}{2},\frac{5}{2})\rrbracket$	$M_3(\frac{3}{2},\frac{5}{2})$
	$\llbracket(-1,-2,-\frac{1}{2},4)\rrbracket$, $\llbracket(-2,-1,-\frac{3}{2},4)\rrbracket$	$M_3(\frac{1}{2},4)$

Table 1.1.1. All non-closed isolated exceptional filling instructions on M_5.

Isolated exceptional filling instructions	Filled manifold
$\llbracket(-2,-2,-2,-1,-7)\rrbracket$ $\llbracket(-1,-2,-3,-1,-7)\rrbracket$, $\llbracket(-1,-2,-3,-5,-3)\rrbracket$, $\llbracket(-1,-2,-6,-2,-3)\rrbracket$, $\llbracket(-1,-3,-1,-6,-3)\rrbracket$, $\llbracket(-1,-4,-1,-6,-2)\rrbracket$, $\llbracket(-1,-2,-2,-5,-4)\rrbracket$	$M_3(-2,-5,-1)$
$\llbracket(-1,-3,-1,-5,-3)\rrbracket$, $\llbracket(-1,-4,-1,-5,-2)\rrbracket$, $\llbracket(-1,-2,-3,-1,-6)\rrbracket$ $\llbracket(-1,-2,-2,-4,-4)\rrbracket$, $\llbracket(-1,-2,-3,-4,-3)\rrbracket$, $\llbracket(-1,-2,-2,-2,-6)\rrbracket$, $\llbracket(-1,-2,-5,-2,-3)\rrbracket$	$M_3(-1,-2,-4)$
$\llbracket(-1,-2,-2,-2,-5)\rrbracket$, $\llbracket(-1,-2,-3,-1,-5)\rrbracket$, $\llbracket(-1,-2,-4,-1,-4)\rrbracket$, $\llbracket(-1,-3,-1,-4,-3)\rrbracket$, $\llbracket(-1,-2,-3,-3,-3)\rrbracket$, $\llbracket(-1,-2,-2,-3,-4)\rrbracket$, $\llbracket(-1,-2,-4,-2,-3)\rrbracket$	$M_3(-1,-3,-2)$

Table 1.1.2. All closed isolated exceptional filling instructions on M_5, part 1/2.

Isolated exceptional filling instructions	Filled manifold
$[\![(-1,-3,-1,-4,-4)]\!]$, $[\![(-1,-2,-2,-3,-5)]\!]$, $[\![(-1,-2,-4,-1,-5)]\!]$, $[\![(-1,-2,-4,-3,-3)]\!]$	$M_3(-3,-1,-3)$
$[\![(-1,-2,-2,-2,-4)]\!]$, $[\![(-1,-2,-3,-1,-4)]\!]$, $[\![(-1,-3,-1,-3,-3)]\!]$, $[\![(-1,-2,-1,-3,-2)]\!]$	$M_3(-1,-2,-2)$
$[\![(-1,-2,-\frac{1}{2},5,3)]\!]$, $[\![(-1,-2,4,5,-\frac{3}{2})]\!]$, $[\![(-1,4,4,-1,-\frac{3}{2})]\!]$	$M_3(5,5,\frac{1}{2})$,
$[\![(\frac{1}{2},-1,-2,\frac{1}{3},\frac{5}{2})]\!]$, $[\![(-1,-2,\frac{3}{2},\frac{5}{2},-\frac{2}{3})]\!]$, $[\![(-1,\frac{1}{2},-1,\frac{1}{3},\frac{3}{2})]\!]$	$M_3(\frac{5}{2},\frac{4}{3},\frac{5}{2})$
$[\![(-1,-2,-\frac{2}{3},4,-3)]\!]$, $[\![(-1,-2,-2,4,-\frac{5}{3})]\!]$, $[\![(-1,-2,3,-1,-\frac{5}{3})]\!]$	$M_3(4,-1,\frac{1}{3})$
$[\![(-1,2,-1,\frac{1}{2},\frac{1}{2})]\!]$, $[\![(-1,\frac{1}{2},3,-1,-\frac{1}{2})]\!]$, $[\![(-2,-1,-\frac{1}{2},\frac{3}{2},3)]\!]$	$M_3(4,\frac{3}{2},\frac{3}{2})$
$[\![(-1,-2,-3,-2,-4)]\!]$, $[\![(-1,-3,-3,-1,-4)]\!]$	$M_3(-2,-2,-2)$
$[\![(-1,-\frac{1}{3},-1,\frac{2}{3},\frac{3}{2})]\!]$, $[\![(-1,-\frac{1}{3},\frac{5}{2},\frac{2}{3},-2)]\!]$	$M_3(\frac{5}{3},\frac{5}{3},\frac{5}{2})$
$[\![(-2,-1,2,4,-\frac{1}{3})]\!]$, $[\![(-1,-2,3,4,-\frac{4}{3})]\!]$	$M_3(4,4,\frac{2}{3})$
$[\![(-1,-\frac{1}{2},-1,\frac{1}{2},\frac{5}{3})]\!]$	$M_3(\frac{3}{2},\frac{3}{2},\frac{8}{3})$
$[\![(-1,-2,\frac{1}{2},\frac{7}{3},\frac{1}{3})]\!]$	$M_3(\frac{3}{2},\frac{7}{3},\frac{7}{3})$
$[\![(-1,-3,-2,-2,-3)]\!]$	m_1
$[\![(-2,-\frac{1}{2},3,3,-\frac{1}{2})]\!]$	m_2
$[\![(-2,-2,-2,-2,-2)]\!]$	m_3

TABLE 1.1.3. All closed isolated exceptional filling instructions on M_5, part 2/2.

Equivalence classes under (1.1)-(1.13)	Equivalence classes under (1.1)-(1.2)
$[\![(1)]\!]$	$[(0)]$, $[(1)]$, $[(\infty)]$
$[\![(-1_0,-1_1)]\!]$	$[(-1_0,-1_1)]$, $[((\frac{1}{2})_0,2_1)]$, $[((\frac{1}{2})_0,(\frac{1}{2})_2)]$ $[(-1_0,2_2)]$
$[\![(-1_0,-2_1,-1_2)]\!]$	$[(-1_0,-2_1,-1_2)]$, $[(2_0,-1_1,(-\frac{1}{2})_2)]$, $[(-1_0,(\frac{1}{2})_1,(\frac{3}{2})_2)]$, $[(-1_0,(\frac{1}{2})_2,3_3)]$, $[(-2_0,2_2,2_3)]$, $[((\frac{1}{2})_0,2_2,(\frac{1}{3})_3)]$, $[((\frac{2}{3})_0,(\frac{1}{2})_2,(\frac{1}{2})_3)]$
$[\![(-2_0,-1_1,-2_2)]\!]$	$[(-2_0,-1_1,-2_2)]$, $[(3_0,(-\frac{1}{2})_1,-1_2)]$, $[(2_0,(\frac{1}{3})_1,-2_2)]$, $[(3_0,(\frac{1}{2})_1,3_2)]$, $[((\frac{1}{2})_0,(\frac{2}{3})_2,(\frac{2}{3})_3)]$, $[((-\frac{1}{2})_0,(\frac{2}{3})_2,2_3)]$, $[((\frac{1}{3})_0,(\frac{1}{2})_2,(\frac{3}{2})_3)]$, $[(-1_0,(\frac{3}{2})_2,(\frac{3}{2})_3)]$
$[\![(-1_0,3_2,-2_4)]\!]$	$[(-1_0,3_2,-2_4)]$, $[((\frac{3}{2})_0,-1_2,(-\frac{1}{2})_3)]$, $[(2_0,-2_2,(-\frac{1}{2})_3)]$, $[(3_0,-1_2,-2_3)]$, $[((\frac{1}{2})_0,(\frac{3}{2})_1,(\frac{2}{3})_2)]$, $[((\frac{1}{3})_0,3_1,(\frac{1}{2})_2)]$, $[((\frac{1}{3})_0,2_1,(\frac{2}{3})_2)]$
$[\![(-1,-2,-2,-1)]\!]$	$[(-1,-2,-2,-1)]$, $[(-\frac{1}{2},-1,3,\frac{1}{2})]$, $[(-2,\frac{1}{2},\frac{3}{2},2)]$, $[(3,\frac{2}{3},2,-1)]$, $[(\frac{3}{2},\frac{1}{3},-1,\frac{1}{2})]$, $[(\frac{2}{3},\frac{1}{2},\frac{1}{2},\frac{2}{3})]$, $[(2,-\frac{1}{2},-\frac{1}{2},2)]$, $[(2,\frac{3}{2},\frac{1}{2},-2)]$, $[(\frac{1}{3},2,2,\frac{1}{3})]$
$[\![(-1,-3,-1,-2)]\!]$	$[(-1,-3,-1,-2)]$, $[(\frac{1}{4},2,\frac{3}{2},\frac{1}{2})]$, $[(-1,3,\frac{1}{2},4)]$, $[(-3,\frac{1}{3},2,2)]$, $[(\frac{3}{4},\frac{2}{3},\frac{1}{2},\frac{1}{2})]$, $[(\frac{4}{3},\frac{1}{2},-1,\frac{1}{3})]$, $[(2,\frac{1}{4},-2,\frac{1}{2})]$, $[(3,-\frac{1}{3},-1,2)]$, $[(\frac{3}{2},\frac{4}{3},\frac{1}{2},-1)]$, $[(2,\frac{3}{4},2,-\frac{1}{2})]$, $[(-1,-\frac{1}{2},4,\frac{1}{2})]$, $[(-\frac{1}{3},-1,2,\frac{2}{3})]$

Table 1.1.4. All non-closed isolated exceptional filling instructions of M_5 mod (1.1)-(1.2), part 1/2.

Equivalence classes under (1.1)-(1.13)	Equivalence classes under (1.1)-(1.2)
$\llbracket(-2,-2,-1,-3)\rrbracket$	$[(-2,-2,-1,-3)]$, $[(\frac{1}{2},\frac{4}{3},\frac{3}{2},\frac{1}{3})]$, $[(3,\frac{1}{2},4,-\frac{1}{2})]$, $[(-2,\frac{1}{4},2,\frac{3}{2})]$, $[(\frac{1}{2},\frac{2}{3},\frac{3}{4},\frac{2}{3})]$, $[(\frac{3}{2},\frac{1}{2},-2,\frac{1}{4})]$, $[(2,\frac{1}{3},-3,\frac{1}{3})]$, $[(3,-1,-\frac{1}{2},4)]$, $[(-1,\frac{1}{3},\frac{3}{2},\frac{4}{3})]$, $[(2,\frac{2}{3},3,-\frac{1}{3})]$, $[(-1,-\frac{1}{3},3,\frac{2}{3})]$, $[(-\frac{1}{2},-\frac{1}{2},2,\frac{3}{4})]$
$\llbracket(-1,-2,\frac{1}{2},\frac{5}{2})\rrbracket$	$[(-1,-2,\frac{1}{2},\frac{5}{2})]$, $[(\frac{1}{2},\frac{5}{2},\frac{1}{3},-1)]$, $[(\frac{2}{5},2,\frac{2}{3},\frac{1}{2})]$, $[(-\frac{3}{2},-\frac{1}{2},2,2)]$, $[(\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{3}{5})]$, $[(-\frac{1}{2},-1,\frac{1}{2},\frac{5}{3})]$, $[(2,\frac{2}{5},3,\frac{1}{2})]$, $[(-2,-\frac{2}{3},-1,2)]$, $[(-1,\frac{1}{2},\frac{5}{3},\frac{2}{3})]$, $[(2,\frac{3}{5},2,\frac{1}{3})]$, $[(-1,-\frac{3}{2},-1,3)]$, $[(-\frac{2}{3},-1,2,\frac{3}{2})]$
$\llbracket(-2,-1,-\frac{1}{2},\frac{5}{2})\rrbracket$	$[(-2,-1,-\frac{1}{2},\frac{5}{2})]$, $[(\frac{1}{2},\frac{5}{3},\frac{2}{3},\frac{1}{3})]$, $[(3,\frac{1}{2},\frac{5}{2},\frac{1}{3})]$, $[(-2,\frac{2}{5},2,-\frac{2}{3})]$, $[(\frac{1}{2},\frac{3}{2},\frac{3}{5},\frac{2}{3})]$, $[(\frac{3}{2},\frac{1}{2},3,\frac{2}{5})]$, $[(2,\frac{1}{3},-\frac{3}{2},-\frac{1}{2})]$, $[(-1,-\frac{1}{2},\frac{3}{2},\frac{5}{3})]$, $[(-1,-\frac{2}{3},3,\frac{3}{2})]$, $[(2,\frac{2}{3},-2,-\frac{2}{3})]$, $[(3,-2,-1,-\frac{3}{2})]$, $[(-\frac{1}{2},\frac{1}{3},2,\frac{3}{5})]$
$\llbracket(-1,-2,-\frac{1}{2},4)\rrbracket$	$[(-1,-2,-\frac{1}{2},4)]$, $[(3,\frac{1}{3},4,\frac{1}{2})]$, $[(\frac{1}{3},\frac{1}{2},\frac{4}{3},\frac{2}{3})]$, $[(\frac{2}{3},\frac{3}{4},2,\frac{1}{3})]$, $[(-2,\frac{1}{4},3,\frac{1}{2})]$, $[(-\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{3}{4})]$, $[(-2,-\frac{1}{3},3,2)]$, $[(-\frac{1}{2},-1,\frac{3}{2},\frac{4}{3})]$, $[(-1,-3,\frac{1}{3},3)]$, $[(-3,-\frac{1}{2},-2,2)]$, $[(-\frac{1}{3},-1,\frac{2}{3},\frac{3}{2})]$, $[(\frac{1}{4},2,\frac{2}{3},\frac{3}{2})]$
$\llbracket(-2,-1,-\frac{3}{2},4)\rrbracket$	$[(-2,-1,-\frac{3}{2},4)]$, $[(\frac{3}{5},\frac{2}{3},\frac{4}{3},\frac{1}{2})]$, $[(\frac{1}{3},4,\frac{1}{2},\frac{5}{3})]$, $[(\frac{2}{3},2,\frac{1}{4},-\frac{2}{3})]$, $[(\frac{1}{2},\frac{3}{2},\frac{3}{4},\frac{2}{5})]$, $[(\frac{1}{4},3,\frac{1}{2},\frac{5}{2})]$, $[(-\frac{1}{2},-3,\frac{3}{5},2)]$, $[(-1,-\frac{1}{2},\frac{5}{2},\frac{4}{3})]$, $[(-1,-\frac{1}{3},\frac{5}{3},\frac{3}{2})]$, $[(2,\frac{2}{5},-2,-\frac{1}{3})]$, $[(-3,-1,-\frac{2}{3},3)]$, $[(-\frac{3}{2},\frac{1}{3},2,\frac{3}{4})]$

TABLE 1.1.5. All non-closed isolated exceptional filling instructions of M_5 mod (1.1)-(1.2), part 2/2.

The following result allows one to establish precisely which filling instructions on M_5 are exceptional, and reduces the task of identifying the corresponding exceptional filling of M_5 to the same task for a filling of either F or of M_3. Descriptions of all $F(\alpha)$ are given in Proposition 1.2.3, and all exceptional fillings $M_3(\alpha)$ are found in Chapter 3.

Theorem 1.1.3. *A filling instruction α for M_5 is exceptional if and only if it contains an isolated exceptional filling instruction. If α is an exceptional filling instruction then $M_5(\alpha)$ is described as m_1, m_2, m_3 or as a filling of F or as a filling of M_3 by one of the following cases:*

Case 1: *Up to the D_5 action generated by (1.1) and (1.2), α contains 1_0 or 0_0, or ∞_0, and*

$$\begin{aligned} M_5\big(1, \tfrac{p}{q}, \tfrac{r}{s}, \tfrac{u}{v}, \tfrac{x}{y}\big) &\cong F\big(\tfrac{p-q}{q}, \tfrac{r}{s}, \tfrac{u}{v}, \tfrac{x-y}{y}\big) \\ M_5\big(0, \tfrac{p}{q}, \tfrac{r}{s}, \tfrac{u}{v}, \tfrac{x}{y}\big) &\cong F\big(-\tfrac{y}{x}, \tfrac{r-s}{r}, \tfrac{q}{q-p}, \tfrac{u-v}{v}\big) \\ M_5\big(\infty, \tfrac{p}{q}, \tfrac{r}{s}, \tfrac{u}{v}, \tfrac{x}{y}\big) &\cong F\big(\tfrac{s}{r}, -\tfrac{x}{y}, -\tfrac{p}{q}, \tfrac{v}{u}\big); \end{aligned}$$

Case 2: *Up to the D_5 action generated by (1.1) and (1.2), α contains $((-1)_0, (-1)_1)$ or $((\frac{1}{2})_0, 2_1)$ or $((\frac{1}{2})_0, (\frac{1}{2})_2)$ or $((-1)_0, 2_2)$, and*

$$\begin{aligned} M_5\big(-1, -1, \tfrac{p}{q}, \tfrac{r}{s}, \tfrac{u}{v}\big) &\cong F\big(-\tfrac{v}{u}, 2, \tfrac{q}{q-p}, \tfrac{r-2s}{s}\big) \\ M_5\big(-\tfrac{1}{2}, 2, \tfrac{p}{q}, \tfrac{r}{s}, \tfrac{u}{v}\big) &\cong F\big(\tfrac{q-2p}{q-p}, \tfrac{v}{u-v}, -2, \tfrac{r}{r-s}\big) \\ M_5\big(\tfrac{1}{2}, \tfrac{p}{q}, \tfrac{1}{2}, \tfrac{r}{s}, \tfrac{u}{v}\big) &\cong F\big(\tfrac{u-v}{v}, 2, \tfrac{r-s}{r}, \tfrac{p-2q}{q}\big) \\ M_5\big(-1, \tfrac{p}{q}, 2, \tfrac{r}{s}, \tfrac{u}{v}\big) &\cong F\big(\tfrac{p}{q}, \tfrac{v+u}{v}, \tfrac{r-s}{s}, -2\big); \end{aligned}$$

Case 3: *Up to the D_5 action generated by (1.1) and (1.2), α contains $((-1)_0, (-1)_2)$ or $((-1)_0, (\frac{1}{2})_2)$ or $((\frac{1}{2})_0, 2_2)$ or $(2_0, 2_2)$ or $((-1)_0, 2_1)$ or $((-1)_0, (\frac{1}{2})_1)$ or $((\frac{1}{2})_0, (\frac{1}{2})_1)$ or $(2_0, 2_1)$, in which case one of the identities (1.3)-(1.13) transforms α into $(-1, \frac{p}{q}, -1, \frac{r}{s}, \frac{u}{v})$, and*

$$M_5((-1, \frac{p}{q}, -1, \frac{r}{s}, \frac{u}{v})) \cong M_3\big(\tfrac{p+2q}{q}, \tfrac{r+s}{s}, \tfrac{u+v}{v}\big);$$

Case 4: *Up to the D_5 action generated by (1.1) and (1.2), α contains $((-1)_0, (-2)_1)$ or $((\frac{1}{2})_0, 3_1)$ or $((\frac{1}{2})_0, (\frac{3}{2})_1)$ or $((\frac{2}{3})_0, 2_1)$ or $((-1)_0, (-\frac{1}{2})_1)$ or $((\frac{1}{3})_0, 2_1)$ or $((\frac{1}{2})_0, (\frac{1}{3})_2)$ or $(2_0, (-\frac{1}{2})_2)$ or $((\frac{1}{2})_0, (\frac{2}{3})_2)$ or $((-1)_0, (\frac{3}{2})_2)$ or $((-1)_0, 3_2)$ or $(2_0, (-2)_2)$, in which case one of the identities 1.3-1.13 transforms α into some $(-1, -2, \frac{p}{q}, \frac{r}{s}, \frac{u}{v})$, and*

$$M_5(-1, -2, \frac{p}{q}, \frac{r}{s}, \frac{u}{v}) \cong M_3\big(\tfrac{p+q}{q}, \tfrac{r}{s}, \tfrac{u+2v}{v}\big);$$

Case 5: *α lies in one of*

$$[\![(-1,-3,-2,-2,-3)]\!], [\![(-2,-\tfrac{1}{2},3,3,-\tfrac{1}{2})]\!], [\![(-2,-2,-2,-2,-2)]\!]$$

in which case $M_5(\alpha)$ is m_1, m_2, m_3 respectively.

1.2. Exceptional fillings of M_5

Our description of the exceptional fillings of M_5 will employ the notation now discussed.

Given integers $p_1, \ldots, p_n, q_1, \ldots, q_n$, with $p_i > 1$, and a compact surface G, we let E denote the surface obtained by removing n open discs from G, and we denote by $b_1, \ldots, b_n$ the n newly introduced boundary circles. When G is orientable $(G, (p_1, q_1), \ldots (p_n, q_n))$ denotes the manifold obtained by fixing an orientation on $E \times S^1$ performing a Dehn filling on each $b_i \times S^1$ along $p_i\mu_i + q_i\lambda_i$ where $\{\mu_i, \lambda_i\} = \{b_i \times \{*\}, \{*\} \times S^1\}$ are oriented so that μ_i, λ_i is a positive basis of $H_1(b_i \times S^1)$ with $b_i \times S^1$ oriented as $\partial(E \times S^1)$.

When G is non-orientable, we fix an orientation on $M = E\tilde{\times}S^1$, the orientable S^1-bundle over E, and choose a cross-section $s : E \to M$ of the fibration $f : M \to E$, fixing λ_i equal to a fibre and $\mu_i = f^{-1}(b_i)$ oriented as above; then $(G, (p_1, q_1), \ldots, (p_n, q_n))$ denotes the manifold obtained by performing a Dehn filling on each boundary $f^{-1}(b_i)$ of $M = E\tilde{\times}S^1$ along $p_i\mu_i + q_i\lambda_i$.

A manifold of the form $(G, (p_1, q_1), \ldots (p_n, q_n))$ is called a *Seifert fibered space*. In our case, G will be either the disc D, the annulus A, the pair of pants P, or the sphere S^2. A Seifert space $M = (G, (p_1, q_1), \ldots, (p_n, q_n))$ is a called a *small Seifert space* when $G = S^2$ and $n \leq 3$, or $G = D^2$ and $n \leq 2$, or $G = A$ and $n \leq 1$, or $G = \mathbb{RP}^2$ and $n \leq 1$, or $G = P$ and $n = 0$, or G is the Möbius band and $n = 0$.

We say that a Seifert space $(S^2, (p_1, q_1), \ldots, (p_n, q_n))$ with $n = 3$ is of *type-Z*, and note that $\big(S^2, (p, q)\big)$ is a lens space which we write as $L(q, p)$.

Remark Note that $\big(S^2, (p, q), (r, s)\big) = L(ps+rq, ps'+r'q)$ where $rs'-sr' = 1$, therefore S^3, $S^2 \times S^1$, and $\mathbb{RP}^3$ are all closed small Seifert manifolds.

When the parameters (p_i, q_i) of a Seifert fibered space $(G, (p_1, q_1), \ldots, (p_n, q_n))$ satisfy $p_i > q_i > 0$ they are called *normalised*. The parameters of a Seifert space $M = (G, (p_1, q_1), \ldots, (p_n, q_n))$ with $p_i, q_i \neq 0$ not necessarily normalised can be normalised using

$$(G, (p_1, q_1), \ldots, (p_n, q_n)) \cong (G, (p_1, q_1), \ldots, (p_i, q_i{+}kp_i), \ldots, (p_n, q_n)), \qquad \text{if } \partial G \neq \varnothing;$$

while if G is closed we introduce a $(1,0)$ filling and use

$$(G,(p_1,q_1),\ldots,(p_n,q_n),(1,0)) \cong (G,(p_1,q_1),\ldots,(p_i,q_i+kp_i),\ldots,(p_n,q_n),(1,-k))$$

to see $M \cong (G,(p_1,q_1'),\ldots,(p_n,q_n'),(1,m))$ with $p_i > q_i' > 0$ for some m, q_i'. In this case we write $M = (G,(p_1,q_1'),\ldots,(p_n,q_n'),m)$. The integer m is called the *Euler number* of M.

The results of this section give a full description of the fillings of M_5. We recall some standard terminology from the theory of 3-manifolds necessary for this description. A surface Σ in a 3-manifold M is said to be *properly embedded* when $\Sigma \cap \partial M = \partial\Sigma$.

Definition A properly embedded surface Σ in a 3-manifold M is said to be *essential* if either

(1) $\pi_1(\Sigma)$ injects into $\pi_1(M)$ and there is no embedding of $\Sigma \times [0,1]$ in M with $\Sigma = \Sigma \times \{0\}$ and$(\partial\Sigma \times [0,1] \cup \Sigma \times \{1\}) \subset \partial M$, or
(2) $\Sigma \cong D^2$ and D^2 is not a component of $\{\partial M \backslash \partial\Sigma\}$, or
(3) $\Sigma \cong S^2$ and B^3 is not a component of $\{M \backslash \Sigma\}$.

A 3-manifold M is *irreducible* if it does not contain an embedded essential S^2. When a manifold contains an essential $S^1 \times S^1$, S^2, or D^2 it is called *toroidal*, *reducible*, *boundary reducible* respectively. When a manifold contains no essential surfaces of non-negative Euler characteristic it is said to be *simple*. A manifold M is said to be *prime* if whenever $M = M_1 \# M_2$ then $M_i = S^3$ for at least one of the i. All manifolds have a prime decomposition (see Hempel [**Hem**] and Kneser [**Kne**]).

Theorem 1.2.1. [Kneser's Theorem] *For every compact orientable 3-manifold M, either $M = S^3$ or there exists prime manifolds $M_1,\ldots,M_m$ with no $M_i = S^3$ and $M = M_1\#\ldots\#M_m$. If $M = M_1\#\ldots\#M_m$ and $M = N_1\#\ldots\#N_n$ are two prime decompositions of a 3-manifold M then $m = n$, and, up to permutation of the summands, for every k there exists homeomorphism $h_k : M_k \to N_k$. If M is oriented then each h_k is orientation-preserving.*

The following is due to Jaco-Shalen and Johannson [**JS**], [**Joh**].

Theorem 1.2.2 (JSJ decomposition). *Every irreducible, boundary irreducible compact orientable 3-manifold M contains a minimal collection $\mathcal{E}$ of disjoint essential tori and annuli such that M cut along $\mathcal{E}$ is a union of Seifert spaces, simple manifolds, and I-bundles over surfaces. This collection is unique up to isotopy.*

For $(G,(p_1,q_1),\dots(p_n,q_n))$ a Seifert space with G an orientable surface with $k \geq 1$ boundary components $b_1, \dots, b_k$, we set E to be G minus n open discs and fix a homology basis (μ_i,λ_i) on each of the $k+n$ boundaries components of $E\times S^1$ as above, with the additional requirement that the $\mu_i \subset b_i$ are oriented as components of ∂G. Given Seifert manifolds X and Y with orientable base surfaces with boundary, and a linear map B of determinant -1, we define $X\bigcup_B Y$ unambiguously to be the quotient manifold $X\bigcup_f Y$ where $f: T\to U$ for T and U arbitrary boundary components of X and Y respectively, and f acts on homology by B with respect to the bases described above. The case $T,U\subset \partial X$, $T\neq U$ is also allowed and we write the quotient manifold as $X\big/_B$.

For the purposes of Proposition 1.2.3 we employ a very flexible notation for Seifert manifolds and lens spaces. We will formally identify an $\varnothing$ slope on F with $\frac{0}{0}$. Our Seifert manifolds $(G,(p_1,q_1),\dots,(p_n,q_n))$ have G orientable, and allow parameters (p_i,q_i) to be non-normalised and to be of the form $(0,1),(1,q_i),(0,0)$; as before G minus n open discs is a surface denoted by E with n newly introduced boundary boundary circles denoted $b_1,\dots,b_n$ and $(G,(p_1,q_1),\dots(p_n,q_n))$ is the manifold obtained by performing a Dehn filling to $E\times S^1$ on each $b_i\times S^1$ along $p_i\mu_i+q_i\lambda_i$ unless $(p_i,q_i)=(0,0)$ in which case no filling along $b_i\times S^1$ is performed. We say that a pair $(p_i,q_i)\neq(0,0)$ with $|p_i|\leq 1$ is *non-genuine*.

For Proposition 1.2.3 and Corollary 1.2.4 we use the following homeomorphisms: $L(1,q)\cong S^3$, $L(0,1)\cong S^2\times S^1$, $L(2,1)\cong \mathbb{RP}^3$, $L(0,0)\cong D^2\times S^1$, and $L(p,q)\cong L(|p|,q')$ where $|p|>q'>0$ and $\operatorname{sgn}(p)q=q' \mod |p|$. So, for example, in Proposition 1.2.3 when $F(\alpha)=L(\pm 2,\pm 1)$ then $F(\alpha)=\mathbb{RP}^3$, and in Corollary 1.2.4 the set $\{(D^2\times S^1)\#L(p,q): p\neq 1\}$ includes $(D^2\times S^1)\#(S^2\times S^1)$, $(D^2\times S^1)\#\mathbb{RP}^3$ and $(D^2\times S^1)\#(D^2\times S^1)$.

Proposition 1.2.3. *Let $\alpha=\left(\frac{p}{q},\frac{r}{s},\frac{u}{v},\frac{x}{y}\right)$ be a filling instruction on F with $\frac{p}{q},\frac{r}{s},\frac{u}{v},\frac{x}{y}\in\mathbb{Q}\cup\{\infty,\varnothing\}$. The following equality holds*

$$F\left(\tfrac{p}{q},\tfrac{r}{s},\tfrac{u}{v},\tfrac{x}{y}\right)=F\left(\tfrac{x}{y},\tfrac{p}{q},\tfrac{r}{s},\tfrac{u}{v}\right), \tag{1.14}$$

$$F\left(\tfrac{p}{q},\tfrac{r}{s},\tfrac{u}{v},\tfrac{x}{y}\right)=F\left(\tfrac{p}{q},\tfrac{x}{y},\tfrac{u}{v},\tfrac{r}{s}\right). \tag{1.15}$$

Moreover, α falls into exactly one of Case 1, Case 2 *or* Case 3 *below, and $F(\alpha)$ is obtained as follows:*

Case 1: *α contains only genuine slopes; in this case*

$$F(\alpha) = X(\tfrac{p}{q}, \tfrac{u}{v}) \bigcup_{\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)} Y(\tfrac{r}{s}, \tfrac{x}{y})$$

where

$$X(\tfrac{a}{b}, \tfrac{c}{d}) = Y(\tfrac{a}{b}, \tfrac{c}{d}) = \big(D, (a,b), (c,d)\big) \text{ if } \tfrac{a}{b}, \tfrac{c}{d} \neq \tfrac{0}{0};$$

$$X(\tfrac{a}{b}, \tfrac{c}{d}) = Y(\tfrac{a}{b}, \tfrac{c}{d}) = \big(A, (c,d)\big) \text{ if } \tfrac{a}{b} = \tfrac{0}{0} \text{ and } \tfrac{c}{d} \neq \tfrac{0}{0};$$

$$X(\tfrac{a}{b}, \tfrac{c}{d}) = Y(\tfrac{a}{b}, \tfrac{c}{d}) = \big(A, (a,b)\big) \text{ if } \tfrac{c}{d} = \tfrac{0}{0} \text{ and } \tfrac{a}{b} \neq \tfrac{0}{0};$$

$$X(\tfrac{a}{b}, \tfrac{c}{d}) = Y(\tfrac{a}{b}, \tfrac{c}{d}) = P \times S^1 \text{ if } \tfrac{a}{b}, \tfrac{c}{d} = \tfrac{0}{0}.$$

Case 2: *α contains at least one slope equal to $\frac{0}{1}$; in this case Equation (1.14) can be used to write $\alpha = \big(\frac{0}{1}, \frac{a}{b}, \frac{c}{d}, \frac{e}{f}\big)$ for appropriate a, b, c, d, e, f and*

$$F(\alpha) = L(c,d) \# L(be+af, a'e+b'f), \text{ where } \begin{cases} ab' - ba' = 1 \text{ if } \frac{a}{b} \neq \frac{0}{0}; \\ a' = b' = 0 \text{ otherwise.} \end{cases}$$

Case 3: *α contains at least one slope equal to $\frac{1}{n}$ and no slope of the form $\frac{0}{1}$; in this case Equations* (1.14) *can be used to write $\alpha = \big(\frac{1}{n}, \frac{a}{b}, \frac{-d}{c+nd}, \frac{e}{f}\big)$ for appropriate n, a, b, c, d, e, f and*

$$F(\alpha) = X(\tfrac{a}{b}, \tfrac{c}{d}, \tfrac{e}{f})$$

where $X(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})$ has the property that

$$X(\tfrac{a}{b}, \tfrac{c}{d}, \tfrac{e}{f}) = X(\tfrac{e}{f}, \tfrac{a}{b}, \tfrac{c}{d}); \tag{1.16}$$

$$X(\tfrac{a}{b}, \tfrac{c}{d}, \tfrac{e}{f}) = X(\tfrac{e}{f}, \tfrac{c}{d}, \tfrac{a}{b}). \tag{1.17}$$

The description of $X(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})$ for all $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$ is given below:

Case 3(i): *Up to Equations (1.16) and (1.17), $(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})$ is of the form $(\frac{g}{h}, \frac{i}{j}, \frac{k}{l})$ or $(\frac{g}{h}, \frac{i}{j}, \frac{0}{0})$ or $(\frac{g}{h}, \frac{0}{0}, \frac{0}{0})$ or $(\frac{0}{0}, \frac{0}{0}, \frac{0}{0})$ where $\frac{g}{h}, \frac{i}{j}, \frac{k}{l}$ are genuine slopes not equal to $\frac{0}{0}$, and*

$$X(\tfrac{g}{h}, \tfrac{i}{j}, \tfrac{k}{l}) = \big(S^2, (g,h), (i,j), (k,l)\big);$$

$$X(\tfrac{g}{h}, \tfrac{i}{j}, \tfrac{0}{0}) = \big(D, (g,h), (i,j)\big);$$

$$X(\tfrac{g}{h}, \tfrac{0}{0}, \tfrac{0}{0}) = \big(A, (g,h)\big);$$

$$X(\tfrac{0}{0}, \tfrac{0}{0}, \tfrac{0}{0}) = P \times S^1.$$

Case 3(ii): *Up to Equations (1.16) and (1.17),* $(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})$ *is of the form* $(\frac{0}{1}, \frac{i}{j}, \frac{k}{l})$ *and*

$$X(\tfrac{a}{b}, \tfrac{c}{d}, \tfrac{e}{f}) = L(i, j)\#L(k, l);$$

Case 3(iii): *Up to Equations (1.16) and (1.17),* $(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})$ *contains no* $\frac{0}{1}$ *slope, and is of the form* $(\frac{1}{n}, \frac{g}{h}, \frac{i}{j})$*; in this case*

$$X(\tfrac{1}{n}, \tfrac{g}{h}, \tfrac{i}{j}) = L\big(gj+i(h+n),\, gi'+(h+n)j'\big), \text{ where } \begin{cases} ab' - ba' = 1 \text{ if } \frac{a}{b} \neq \frac{0}{0}; \\ a' = b' = 0 \text{ otherwise.} \end{cases}$$

Corollary 1.2.4. *The complete list of the exceptional fillings of* M_5 *is given by:*

Toroidal manifolds:

$$X \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} Y \text{ for } X, Y \in \big\{P\times S^1, \big(A, (p, q)\big), \big(D, (p, q), (r, s)\big) : p, q, r, s \in \mathbb{Z}, |p|, |r| > 1\big\},$$

$$\big(D, (2, 1), (2, 1)\big) \bigcup_B \big(D, (2, 1), (3, 1)\big), \text{ where } B \in \left\{ \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 5 \\ 1 & 4 \end{pmatrix} \right\}$$

$$T/_{\begin{pmatrix} n & 1 \\ -1 & 0 \end{pmatrix}} \text{ where } n \in \mathbb{Z}, \qquad \big(A, (p, q)\big)/_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \text{ where } p > |q| > 0, \qquad \big(A, (2, -1)\big)/_{\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}},$$

Reducible manifolds:

$$L(p, q)\#L(r, s), \qquad (D^2\times S^1)\#L(p, q), \qquad (D^2\times S^1)\#(D^2\times S^1), \text{ where } |p|, |r| \neq 1$$

Seifert spaces:

All small Seifert spaces.

Moreover, for every manifold $m \notin \{T/_{\begin{pmatrix} n & 1 \\ -1 & 0 \end{pmatrix}} : n = -1, 0, 1, 2, 3\}$ *in this list there exists some hyperbolic filling instruction* α *and a slope* $\frac{p}{q}$ *on a boundary component of* $M_5(\alpha)$ *such that* $M_5(\alpha)(\frac{p}{q}) = m$.

1.3. Exceptional slopes on $M_5(\alpha)$

We begin with some standard notation used in the description of the set of exceptional slopes on a fixed boundary component of a hyperbolic manifold.

If X is a hyperbolic 3-manifold with boundary consisting of tori and τ is a fixed boundary component of ∂X then the set of exceptional slopes on τ is denoted by $E_\tau(X)$ and the cardinality of $E_\tau(X)$ by $e_\tau(X)$. When α is a filling instruction on M_5 and $X = M_5(\alpha)$, our discussion of exceptional slopes on X will always assume α does not contain a slope on the fourth component and take τ to be the fourth component of ∂M_5 to write $E(M_5(\alpha))$ and $e(M_5(\alpha))$ unambiguously. In addition, to describe the slopes in $E(M_5(\alpha))$ we use the homology basis coming from M_5.

To describe $E(M_5(\alpha))$ we introduce the following definition:

Definition Let α be a filling instruction on a manifold X. We say that α *factors through* a manifold Y if there exists some filling instruction $\beta \subset \alpha$ such that $Y = X(\beta)$.

Note if α is exceptional for X and factors through a hyperbolic Y with $Y = X(\beta)$, then $\alpha \backslash \beta$ is exceptional for Y. Denote the exteriors of the 4-chain links 4CL and L of Figure 1.3.1 by M_4 and L_4 respectively. We remark that both 4CL and L are hyperbolic links and emphasize that 4CL is not equivalent to the link M4CL from Figure 1.1.1.

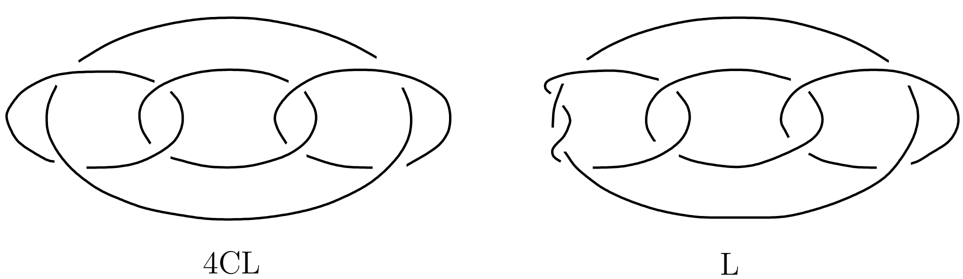


Figure 1.3.1. 4CL and L; chain links with 4 components.

Corollary 1.3.1. *Let α be a hyperbolic filling instruction on M_5 containing at least one $\varnothing$ and not factoring through M_4. Then either $E(M_5(\alpha)) = \{0, 1, \infty\}$ or α factors through the exterior of L and an element of $[\![\alpha]\!]$ is found in Table 1.3.1 or 1.3.2.*

Corollary 1.3.2. *Let α be a hyperbolic filling instruction on M_5 containing at least one $\varnothing$ and factoring through M_4, but not factoring through M_3. Then either $E(M_5(\alpha)) = \{-1,0,1,\infty\}$ or an element of $[\![\alpha]\!]$ is found in Table 1.3.3.*

α	$E(M(\alpha))$
$(-2,-\frac{1}{2},3,3),(-2,\frac{3}{2},\frac{3}{2},-2)$	$\{-1,-\frac{1}{2},0,1,\infty\}$
$(-2,-3,-\frac{1}{2},-2),(-2,-\frac{1}{3},3,\frac{2}{3}),(-2,-\frac{1}{2},3,\frac{2}{3}),$	$\{-1,2,0,1,\infty\}$
$(-2,-2,-2,-2),(-2,\frac{1}{3},3,\frac{1}{3})$	$\{-2,-1,0,1,\infty\}$
$(-2,\frac{1}{3},\frac{3}{2},\frac{3}{2}),(-2,-2,\frac{1}{3},3)$	$\{\frac{1}{3},\frac{1}{2},0,1,\infty\}$
$(-2,-\frac{1}{2},-2,\frac{3}{2})$	$\{\frac{3}{2},2,0,1,\infty\}$
$(-2,\frac{p}{q},3,\frac{u}{v}),(-2,\frac{p}{q},\frac{r}{s},-2)$	$\{-1,0,1,\infty\}$
$(-2,\frac{3}{2},\frac{3}{2},\frac{u}{v}),(-2,\frac{p}{q},\frac{5}{2},-\frac{1}{2}),(-2,-2,\frac{r}{s},-3),$ $(-2,\frac{3}{2},\frac{r}{s},-\frac{1}{2}),(-2,-\frac{1}{2},4,\frac{u}{v}),(-2,\frac{p}{q},4,-\frac{3}{2})$ $(-2,3,\frac{r}{s},-\frac{3}{2})$	$\{-1,0,1,\infty\}$
$(-2,\frac{p}{q},\frac{1}{3},3),(-2,\frac{1}{3},\frac{r}{s},\frac{3}{2}),$ $(-2,\frac{1}{4},\frac{r}{s},\frac{3}{2}),(-2,\frac{2}{3},\frac{2}{3},\frac{u}{v}),(-2,\frac{p}{q},\frac{2}{3},\frac{3}{2})$	$\{\frac{1}{2},0,1,\infty\}$
$(-2,-2,-\frac{1}{2},\frac{u}{v}),(-2,-\frac{1}{2},-2,\frac{u}{v}),(-2,\frac{p}{q},-2,\frac{1}{3}),$ $(-2,-\frac{1}{2},\frac{r}{s},\frac{2}{3})$	$\{2,0,1,\infty\}$

Table 1.3.1. α not factoring through M_4 with $e\big(M_5(\alpha)\big) > 3$, part 1/2.

α	$E(M(\alpha))$
$(-2,-\frac{1}{3},3,\frac{2}{3}),(-2,-\frac{2}{3},-2,\frac{2}{3}),(-2,-2,-\frac{1}{3},3),$ $(-2,-\frac{1}{3},-2,\frac{2}{5}),(-2,-3,-\frac{1}{2},-2),(-2,-\frac{1}{2},-\frac{3}{2},\frac{1}{3})$ $(-2,\frac{1}{3},-3,\frac{1}{3}),(-2,-\frac{1}{2},-3,\frac{1}{3}),(-2,-\frac{1}{2},-3,\frac{3}{5})$	$\{2,0,1,\infty\}$
$(-2,4,5,-\frac{3}{2}),(-2,-\frac{1}{2},5,3),(-2,3,4,-\frac{4}{3}),(-2,3,\frac{3}{2},-\frac{1}{2})$ $(-2,-2,4,-\frac{5}{3}),(-2,-\frac{2}{3},4,-3),(-2,3,\frac{1}{3},-3),(-2,\frac{5}{3},\frac{3}{2},-\frac{1}{2})$ $(-2,\frac{2}{3},\frac{5}{2},-\frac{1}{3}),(-2,\frac{3}{2},\frac{5}{3},-\frac{1}{3}),(-2,\frac{4}{3},\frac{3}{2},\frac{1}{3}),(-2,-2,-2,-4)$ $(-2,-3,-2,-3),(-2,-2,-2,-5),(-2,-4,-2,-3),$ $(-2,-2,-3,-5),(-2,-4,-3,-3),(-2,-2,-2,-6),$ $(-2,-5,-2,-3),(-2,-2,-4,-4),(-2,-3,-4,-3),$ $(-2,-2,-2,-7),(-2,-6,-2,-3),(-2,-2,-5,-4)$ $(-2,-3,-5,-3),(-2,-2,-3,-5),(-2,-4,-3,-3),$ $(-2,-3,-2,-4),(-2,\frac{4}{3},\frac{7}{3},-\frac{1}{2}),(-2,-3,-5,-3),$ $(-2,-3,-3,-3),(-2,-2,-2,-4),(-2,-2,-3,-4),$ $(-2,\frac{3}{2},\frac{5}{2},-\frac{2}{3})$	$\{-1,0,1,\infty\}$
$(-2,-2,\frac{1}{4},3),(-2,\frac{2}{5},\frac{3}{4},\frac{3}{2}),(-2,\frac{1}{5},\frac{4}{3},\frac{3}{2}),(-2,\frac{1}{4},\frac{2}{3},\frac{5}{3})$ $(-2,\frac{1}{5},\frac{3}{2},\frac{3}{2}),(-2,3,\frac{1}{3},4),(-2,\frac{1}{4},\frac{3}{2},\frac{4}{3}),(-2,\frac{1}{3},\frac{3}{2},\frac{4}{3})$ $(-2,-2,\frac{1}{5},3),(-2,\frac{2}{3},\frac{3}{5},\frac{3}{2}),(-2,\frac{1}{6},\frac{3}{2},\frac{3}{2}),(-2,\frac{1}{7},\frac{3}{2},\frac{3}{2})$ $(-2,\frac{1}{8},\frac{3}{2},\frac{3}{2}),(-2,\frac{1}{5},\frac{6}{5},\frac{3}{2}),(-2,\frac{3}{8},\frac{3}{4},\frac{3}{2}),(-2,\frac{1}{3},\frac{2}{3},\frac{5}{3}),$ $(-2,\frac{1}{3},\frac{1}{3},3),(-2,\frac{2}{3},\frac{3}{4},\frac{2}{3}),(-2,\frac{2}{3},\frac{1}{3},3),(-2,\frac{1}{3},\frac{2}{3},\frac{5}{3}),$ $(-2,\frac{3}{5},\frac{2}{3},\frac{4}{3})$	$\{\frac{1}{2},0,1,\infty\}$

Table 1.3.2. α not factoring through M_4 with $e\big(M_5(\alpha)\big)>3$, part 2/2.

α	$E(M(\alpha))$
$(-1,-3,-2,-2)$	$\{-3,-2,-1,0,1,\infty\}$
$(-1,-3,-\frac{1}{2},-2)$, $(-1,-\frac{1}{3},\frac{5}{2},\frac{2}{3})$	$\{-2,2,-1,0,1,\infty\}$
$(-1,-\frac{1}{3},4,\frac{2}{3})$	$\{3,2,-1,0,1,\infty\}$
$(-1,3,\frac{2}{3},4)$	$\{-\frac{1}{2},-\frac{1}{3},-1,0,1,\infty\}$
$(-1,\frac{1}{3},\frac{4}{3},\frac{4}{3})$	$\{\frac{1}{2},\frac{1}{3},-1,0,1,\infty\}$
$(-1,-\frac{1}{3},\frac{r}{s},\frac{2}{3}),(-1,\frac{p}{q},-2,\frac{1}{3}),(-1,-\frac{2}{3},-2,\frac{u}{v})$	$\{2,-1,0,1,\infty\}$
$(-1,-3,\frac{r}{s},-2),(-1,-\frac{3}{2},4,\frac{u}{v}),(-1,\frac{p}{q},4,-\frac{1}{2})$	$\{-2,-1,0,1,\infty\}$
$(-1,3,5,-\frac{1}{2}),(-1,-3,4,-\frac{2}{3}),(-1,-3,-5,-3),$ $(-1,-\frac{5}{3},4,-2),(-1,-3,-3,-4),(-1,-4,-5,-2),$ $(-1,-4,-2,-2),(-1,-3,-2,-3),(-1,-5,-3,-2),$ $(-1,-3,-2,-4),(-1,-5,-2,-2),(-1,-3,-3,-3),$ $(-1,-7,-2,-2),(-1,-3,-2,-5),(-1,-6,-2,-2),$ $(-1,-4,-4,-2),(-1,-3,-2,-6),(-1,-4,-2,-3),$	$\{-2,-1,0,1,\infty\}$
$(-1,-\frac{2}{3},-3,\frac{4}{3}),(-1,\frac{1}{3},-3,\frac{1}{3}),(-1,-\frac{3}{5},-2,\frac{2}{3}),$ $(-1,-\frac{1}{3},-2,\frac{2}{5}),(-1,-\frac{1}{3},5,\frac{3}{4}),(-1,-\frac{1}{4},7,\frac{2}{3}),$ $(-1,-\frac{1}{3},4,\frac{4}{5}),(-1,-\frac{1}{4},4,\frac{3}{4}),(-1,-\frac{1}{5},5,\frac{2}{3})$ $(-1,-\frac{1}{4},5,\frac{2}{3}),(-1,-\frac{1}{3},4,\frac{6}{7}),(-1,-\frac{1}{3},7,\frac{3}{4}),$ $(-1,-\frac{1}{7},4,\frac{2}{3}),(-1,-\frac{1}{3},4,\frac{5}{6}),(-1,-\frac{1}{3},6,\frac{3}{4}),$ $(-1,-\frac{1}{4},6,\frac{2}{3}),(-1,-\frac{1}{3},5,\frac{4}{5})$	$\{2,-1,0,1,\infty\}$

TABLE 1.3.3. α factoring through M_4 but not through M_3 with $e\big(M_5(\alpha)\big) > 4$

CHAPTER 2

Proofs of the classification results

In this chapter we present the proofs of all the results in Chapter 1. After proving Theorem 1.0.7 below, we begin in Section 2.1 with some important results from the literature that lead us to the proofs of Theorems 1.1.1 and 1.1.3. Section 2.2 begins with some standard identities on Seifert and graph manifolds in Lemma 2.2.1, that we use to prove Proposition 1.2.3 and Corollary 1.2.4. Finally, in Section 2.3, we see that Corollaries 1.3.1 and 1.3.2 are straightforward (but long) consequences of the previously proved results.

Our proofs rely on a computer program written by Jeffrey Weeks called SnapPea. This software performs various calculations on 3-manifolds, that can be fed to SnapPea as link complements or as triangulations (see [**We1**], [**CDW**], and also [**We2**]). The main features of the program are the following two functions; firstly, SnapPea takes a link as input and returns an ideal triangulation of the link complement, and secondly, SnapPea takes an ideal triangulation of a cusped hyperbolic manifold as input and returns the canonical decomposition of the manifold (see [**EP**] for details on the canonical decomposition and [**BP**] for full details on the material on hyperbolic manifolds not presented in this thesis). One of the additional features of the program uses the canonical decomposition of a cusped hyperbolic manifold to compute its symmetry group and the induced action on filling instructions (see [**We1**], [**CDW**] and documentation therein for a full description of the program's capabilities). This last feature easily allows one to establish Theorem 1.0.7:

Proof of Theorem 1.0.7. SnapPea computes the symmetry group of M_5 to be $S_5 \times \mathbb{Z}/2\mathbb{Z}$ and determines its action on slopes (see [**We1**]). The maps (1.1)-(1.13) come directly from SnapPea.

Each element of the symmetry group of M_5 acts on the set of boundary components of M_5, and on the set of filling instructions $\mathcal{S}^5$. We will first demonstrate that the action on the set of boundary components of M_5 generated by the maps

(1.1)-(1.13) is that of the full S_5, and then we will use this fact to conclude that the action on $\mathcal{S}^5$ induced by the symmetry group of M_5 is generated by the maps (1.1)-(1.13).

No two of (1.3)-(1.13), considered as permutations of the boundary components, are equal up to the D_5 action generated by (1.1) and (1.2). Thus, each of (1.3)-(1.13) corresponds to a representative element of a distinct left coset of D_5 in S_5. Since there are $\frac{5!}{10} = 12$ such cosets, and our list of maps (1.3)-(1.13) consists of 11 items, the symmetries of M_5 corresponding to (1.1)-(1.13) generate the full S_5 action.

We recall that SnapPea computes the order of the symmetry group of M_5 to be 240; the generator of the remaining $\mathbb{Z}/2\mathbb{Z}$ corresponds to a strong involution of 5CL with a trivial action on $\mathcal{S}^5$. Thus the equivalence relation on $\mathcal{S}^5$ induced by the action of the symmetry group of M_5 is generated by the maps (1.1)-(1.13). $\square$

2.1. Classification

In this section we prove Theorem 1.1.1, Proposition 1.1.2, and Theorem 1.1.3. To do so we recall some well-known results from the literature.

A *surgery diagram* is a link diagram with a surgery instruction on each link component. The following is a well-known result found in Rolfsen's text on knots and links (see [**Rol**]) and will be extremely useful.

THEOREM 2.1.1. *Given surgery diagrams* $(L, (\frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k}))$, $(L', (\frac{p'_1}{q'_1}, \ldots, \frac{p'_k}{q'_k}))$ *as in Figure 2.1.1, we have a homeomorphism* $L(\frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k}) \cong L'(\frac{p'_1}{q'_1}, \ldots, \frac{p'_k}{q'_k})$ *where*

$$\frac{p'_i}{q'_i} = \begin{cases} \frac{p_i}{q_i+p_i} & \text{if } i = 1; \\ \frac{p_i+q_i(\mathrm{lk}(l_1,l_i))^2}{q_i} & \text{otherwise.} \end{cases}$$

Remark The move transforming (L, f) into (L', f') is called a 'positive twist' or a 'right-handed twist', while the inverse move is called a 'negative twist' or a 'left-handed twist'.

Definition We will say that two surgery diagrams (L, f) and (L', f') are equivalent if $L(f)$ and $L'(f')$ are homeomorphic by an orientation-preserving homeomorphism.

Remark If a component of a surgery diagram bears the slope ∞ then the component can be removed without affecting the manifold resulting from the surgery. So, if a surgery diagram (L, α) contains the trivial knot with slope $\frac{1}{k}$, then applying

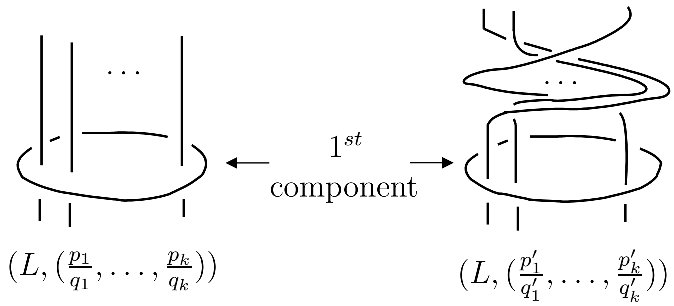


Figure 2.1.1. A positive twist on the 1^{st} component of L.

Theorem 2.1.1 k times, ones sees that (L, α) is equivalent to some surgery diagram (L', α') where L' has one fewer components. This fact will be used throughout the thesis to realise many surgeries on 5CL as surgeries on 3CL.

Example 2.1.2. In Figure 2.1.2 we show the links 5CL, 4CL, 3CL, and M4CL first introduced in Figures 1, 1.1.1 and 1.3.1 (one easily sees that the diagrams in Figure 2.1.2 are related to those in Figures 1, 1.1.1 and 1.3.1 by Reidemeister moves). We then twist along the components highlighted in Figure 2.1.2 and use Theorem 2.1.1 to get:

$$M_5\big(\tfrac{p}{q}, \tfrac{r}{s}, \tfrac{u}{v}, 1, \tfrac{x}{y}\big) \cong F\big(\tfrac{p}{q}, \tfrac{r}{s}, \tfrac{u-v}{v}, \tfrac{x-y}{y}\big)$$

$$M_5\big(\tfrac{p}{q}, -1, \tfrac{r}{s}, \tfrac{u}{v}, \tfrac{x}{y}\big) \cong M_4\big(\tfrac{p+q}{q}, \tfrac{r+s}{s}, \tfrac{u}{v}, \tfrac{x}{y}\big)$$

$$M_4\big(\tfrac{p}{q}, \tfrac{r}{s}, -1, \tfrac{u}{v}\big) \cong M_3\big(\tfrac{p}{q}, \tfrac{r+s}{s}, \tfrac{u+v}{v}\big).$$

We now present the Geometrization Theorem. The non-closed case and the closed case for Haken manifolds are due to Thurston [**Th1**] and the general closed case is due to Perelman [**Pe1**], [**Pe2**], [**Pe3**].

Theorem 2.1.3. [The Geometrization Theorem] *A 3-manifold M is simple if and only if $M \backslash \partial M$ admits a complete hyperbolic structure, or M is a small closed Seifert space, or M is B^3.*

Geometrization implies that $M_5(f)$ is non-hyperbolic if and only if $M_5(f)$ contains an essential torus, sphere, annulus, disc, or is a closed small Seifert space. We use this fact throughout the proofs of Theorem 1.1.1-1.1.3. Also of enormous importance to the proof of Theorem 1.1.1 is the construction of a finite set of filling instructions that contain all isolated exceptional filling instructions. The main

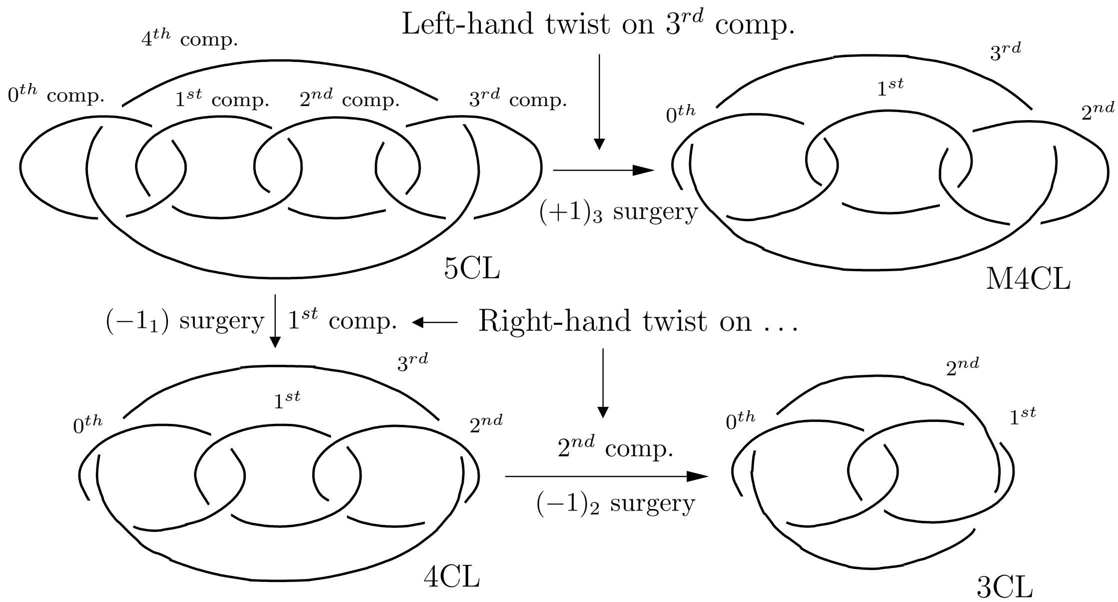


FIGURE 2.1.2. Theorem 2.1.1 applied to the chain links from Chapter 1.

step in creating such a set uses the 6-Theorem of Lackenby ([**Lac**], see also [**Ago**] and [**BH**]).

Let M be an orientable compact manifold with toroidal boundary components $T_1, \ldots, T_k$ and hyperbolic $\text{int}(M)$. An embedded surface $H = H_1 \sqcup \cdots \sqcup H_k$ parallel to $\bigcup T_i$ that lifts to a horosphere in $\mathbb{H}^3$ is called a horospherical cusp section (see [**Th2**] or [**BP**] for details). The *H-length* of a simple closed curve γ on T_i, denoted by $l_H(\gamma)$, is defined to be the minimum length of all simple closed curves in H_i parallel to γ.

Let $\text{int}(M)$ be the complement of a hyperbolic link L with k components in S^3, and $p : \mathbb{H}^3 \to \text{int}(M)$ be the universal cover of $\text{int}(M)$. Let $C_i = N(L_i) \backslash L_i$ be such that $\text{cl}(p^{-1}(C_1)) \cup \cdots \cup \text{cl}(p^{-1}(C_k))$ is a set of closed horoballs with disjoint interiors, where $N(L_i)$ denotes a neighbourhood of L_i. The $C_1, \ldots, C_n$ are called *simultaneously maximal cusps* of $S^3 \backslash K$ when each C_i has the same volume, and that volume is maximal among all possible such configurations. When a cusp section corresponds to the boundary of simultaneously maximal cusps in $S^3 \backslash K$ we say that the cusp section is *maximal*. See [**EP**] for full details.

The Geometrization Theorem in conjunction with the statement of the 6-Theorem in [**Lac**] implies:

THEOREM 2.1.4. [6-Theorem] *Let M be a hyperbolic manifold with toroidal boundary components $T_0, \ldots, T_k$ and slopes $\alpha_0, \ldots, \alpha_k$ with α_i on T_i, where α_i is also permitted to be $\varnothing$. If the H-length of each α_i not equal to $\varnothing$ exceeds 6 for some horospherical cusp section $H = H_1 \sqcup \cdots \sqcup H_k$ then $M(\alpha)$ is hyperbolic.*

We finally recall that manifolds obtained by gluing copies of $P \times S^1$ and $D^2 \times S^1$ by homeomorphisms along some components of their boundaries are called *graph manifolds* and remark that all graph manifolds are known to be non-hyperbolic.

PROOF OF THEOREM 1.1.1. Let T be a component of a horospherical cusp section in a hyperbolic manifold parallel to a boundary component B. We know that T is isometric to a quotient of $\mathbb{C}$ by some lattice $\mathcal{L}$ where the generators of the lattice correspond to the elements of a basis of $H_1(T)$. Let A denote the area of $\mathbb{C}/\mathcal{L}$.

After rotation and dilation by λ we have $\mathcal{L} = \langle 1, x+iy \rangle$ with $y > 0$, whence $\lambda = \sqrt{\frac{y}{A}}$. The complex number $z = x + iy$ is called the *shape* of the cusp. When $H_1(T)$ is given the basis which acts on $\mathbb{C}$ according to the translations $z \mapsto z + \sqrt{\frac{A}{y}}$ and $z \mapsto z + (x+iy)\sqrt{\frac{A}{y}}$, the *length* of a curve on T, parallel to a $\frac{p}{q}$-slope on B, denoted by $l(\frac{p}{q})$, is a rescaling of the Euclidean length of the line segment between 0 and $p + q(x+iy)$ in $\mathbb{C}$.

Our dilation factor scales the length of line segments in $\mathbb{C}/\mathcal{L}$ by $\sqrt{\frac{y}{A}}$. This means that

$$l(\tfrac{p}{q})^2 = \frac{A}{y}((p+xq)^2 + (yq)^2). \tag{2.1}$$

We now construct 5 sets $\mathcal{E}_0, \ldots, \mathcal{E}_4$ with the property that every isolated filling instruction of M_5 is contained in some $\mathcal{E}_i$.

We recall that the boundary components of M_5 are labelled from 0 to 4, that $\mathcal{S}$ denotes $\mathbb{Q} \cup \{\infty, \varnothing\}$ and that a slope α on the k-th boundary component B_k of M_5 with $[\alpha] = p[\mu_k] + q[\lambda_k]$ in $H_1(B_k)$ is identified with an element of $\mathcal{S}$ by writing $(\frac{p}{q})_k$ (where (μ_k, λ_k) is the standard (meridian, longitude) basis for $H_1(B_k)$).

We now define $\mathcal{E}_i$ recursively.

- Let $H(n_0)$ be the maximal cusp section in M_5 corresponding to simultaneously maximal cusps in $S^3 \backslash 5CL$. We define

$$\mathcal{C}_0 = \{s_{n_0} \in \mathcal{S} : l_{H(n_0)}(s_{n_0}) \leq 6 \text{ and } n_0 \in \{0, \ldots, 4\}\}.$$

- We use SnapPea to check if $M_5(s_{n_0})$ is hyperbolic for each $s_{n_0} \in \mathcal{C}_0$ and define
$$\mathcal{E}_0 = \{s_{n_0} \in \mathcal{C}_0 : M_5(s_{n_0}) \text{ is non-hyperbolic, and } n_0 \in \{0,\ldots,4\}\}.$$
- For $s_{n_0} \in \mathcal{C}_0 \backslash \mathcal{E}_0, \ldots,\ s_{n_{k-1}} \in \mathcal{C}_{k-1} \backslash \mathcal{E}_{k-1}$ we define
$$\mathcal{C}_k(s_{n_0},\ldots,s_{n_{k-1}}) = \{s_{n_k} \in \mathcal{S} : l_{H(n_k)}(s_{n_k}) \leq 6, \text{ and } n_k \in \{0,\ldots,4\}\}$$
where $H(n_k)$ is the maximal cusp section in $M_5(s_{n_0},\ldots,s_{n_{k-1}})$.
- We use SnapPea to check if $M_5(s_{n_0},\ldots,s_{n_{k-1}},s_{n_k})$ is hyperbolic for each $s_{n_k} \in \mathcal{C}(s_{n_0},\ldots,s_{n_{k-1}})$ and define
$$\mathcal{E}(s_{n_0},\ldots,s_{n_{k-1}}) = \{s_{n_k} \in \mathcal{C}(s_{n_0},\ldots,s_{n_{k-1}}) : M_5(s_{n_0},\ldots,s_{n_{k-1}},s_{n_k}) \text{ non-hyperbolic}\}.$$
- We define $\mathcal{C}_k = \bigcup \mathcal{C}_k(s_{n_0},\ldots,s_{n_{k-1}})$ and $\mathcal{E}_k = \bigcup \mathcal{E}_k(s_{n_0},\ldots,s_{n_{k-1}})$ where the unions are over all $\{s_{n_0},\ldots,s_{n_{k-1}}\}$ in $\mathcal{C}_{k-1} \backslash \mathcal{E}_{k-1}$.

The key observation is that the 6-Theorem allows us to conclude that if $\alpha = \{s_{n_0},\ldots,s_{n_l}\}$ is an exceptional filling instruction on M_5 then some $s_{n_k} \in \alpha$ is in $\mathcal{E}_k(s_{n_0},\ldots,s_{n_{k-1}})$, so $\bigcup \mathcal{E}_k$ contains all isolated exceptional fillings. For example, if $\alpha = \{s_{n_0}, s_{n_1}, s_{n_2}\}$ is exceptional then $s_{n_0} \in \mathcal{E}_0$, or $s_{n_0} \in \mathcal{C}_0 \backslash \mathcal{E}_0$ and $s_{n_1} \in \mathcal{E}(s_{n_0})$, or $s_{n_0} \in \mathcal{C}_0 \backslash \mathcal{E}_0$ and $s_{n_1} \in \mathcal{C}_1(s_{n_0}) \backslash \mathcal{E}_1(s_{n_0})$ and $s_{n_2} \in \mathcal{E}_2(s_{n_0}, s_{n_1})$.

We now look at the set $\mathcal{E}_0$. Using SnapPea we see that the shapes of the toroidal components of the maximal cusp section of M_5 are the same. We use SnapPea to compute the cusp shape and cusp area A of a toroidal component of a cusp section corresponding to simultaneous maximal cusps in M_5. The result is $A = \sqrt{3}$ and $z = \frac{1}{2} + \frac{\sqrt{3}}{2} i$. Using (2.1) we then easily get
$$\mathcal{C}_0 = \left\{\infty, -3, -2, -1, -\tfrac{1}{2}, -\tfrac{1}{3}, 0, \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}, 1, \tfrac{4}{3}, \tfrac{3}{2}, 2, 3, 4\right\}.$$
Using the action of the symmetry group of M_5 we may assume that the slope from $\mathcal{C}_0$ is on the cusp about the 0-th component of 5CL, and we can actually replace $\mathcal{C}_0$ by $\mathcal{F}_0 = \{1, -1, -2, -3\}$.

Of course the large cardinalities of $\mathcal{C} = \bigcup \mathcal{C}_k$ and $\mathcal{E} = \bigcup \mathcal{E}_k$ make constructing $\mathcal{C}$ and $\mathcal{E}$ almost impossible by hand. A program was written to use a scriptable Python version of SnapPea for the analysis in [**MP**], and we use a modified version of the same program to construct $\mathcal{C}$ and $\mathcal{E}$ (see [**Mar**]).

We set $\mathcal{F}_k = \mathcal{E}_k \mod (1.1) - (1.13)$ which is easily computed using a simple Python routine (see Theorem 1.0.7), and $\mathcal{F} = \bigcup \mathcal{F}_k$. The cardinalities of $\mathcal{E}$, $\mathcal{F}$ and each $\mathcal{E}_k$, $\mathcal{F}_k$ are shown in Table 2.1.1.

Number of slopes	1	2	3	4	5	Total
Cardinality of $\mathcal{E}_k$	1	3	32	96	858	990
Cardinality of $\mathcal{F}_k$	1	1	5	10	201	218
Cardinality of E_k	1	1	3	7	62	74

Table 2.1.1. The cardinalities of the exceptional instructions $\mathcal{E}$, the exceptional instructions mod (1.1)-(1.13) $\mathcal{F}$, and the isolated exceptional instructions E contained in $\mathcal{F}$.

Using Example 2.1.2 we see that α factors through M_3 when it contains a filling instruction in $[\![(-1_0, -1_2)]\!]$ or $[\![(-1_0, -2_1)]\!]$. We use this to write an equally simple Python routine to show that the only α's in $\mathcal{F}$ with α not factoring through M_3 have $[\![\alpha]\!]$ contained in the following set;

$$\mathcal{H} = \{[\![(1_0)]\!], [\![(-1_0, -1_1)]\!], [\![(-1, -3, -2, -2, -3)]\!], [\![(-2, -\tfrac{1}{2}, 3, 3, -\tfrac{1}{2})]\!], \\ [\![(-2, -2, -2, -2, -2)]\!]\}.$$

We now show that each α in $\mathcal{H}$ is exceptional by showing that $M_5(\alpha)$ is a graph manifold. For $\alpha \in [\![(1_0)]\!]$ we saw in Example 2.1.2 that $M_5(\alpha) = F$, the exterior of M4CL.

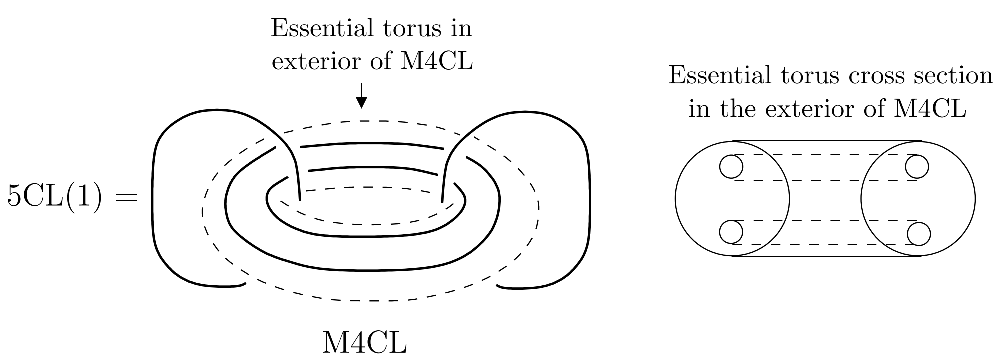


Figure 2.1.3. An exceptional torus in the exterior of M4CL.

Figure 2.1.3 highlights an exceptional torus T in F, and it is clear that T separates F into two copies of $P \times S^1$ which are glued together by identifying a boundary component $\gamma \times S^1$ of one $P \times S^1$ to the other $P \times S^1$ with a horizontal loop $\gamma \times \{*\}$ in the former identified to a fibre $\{*\} \times S^1$ in the latter. Thus F is homeomorphic to $P \times S^1 \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} P \times S^1$.

If $\alpha \in [\![(-1_0, -1_1)]\!]$ then by Theorem 2.1.1 and Theorem 1.0.7 we have

$$M_5(\alpha) = M_5\left(-1_0, -1_1\right) = M_4(0_0) = M_5\left(0_0, -1_2\right) = M_5\left(1_0, 2_2\right) = F\left(2\right).$$

For $\alpha \in [\![(-1, -3, -2, -2, -3)]\!], [\![(-2, -\frac{1}{2}, 3, 3, -\frac{1}{2})]\!], [\![(-2, -2, -2, -2, -2)]\!]$ we use Matveev's computer program *Recognizer* (see [**Ma1**] and [**Mat**]) to show that $M_5(\alpha)$ are homeomorphic respectively to the manifolds m_1, m_2, m_3 defined in Section 1.1. Thus for each $\alpha \in \mathcal{H}$ we see that α is exceptional. It is easy to check that each $\alpha \in \mathcal{H}$ is actually isolated.

For $\alpha \in \mathcal{F} \backslash \mathcal{H}$ we can use the analysis of [**MP**] (see Chapter 3) to determine if α is exceptional. More precisely, $M_5(\alpha) = M_3(\beta)$ for some filling instruction β, and α is an exceptional filling instruction on M_5 if and only if β is an exceptional filling instruction on M_3. The filling instruction β is explicitly determined for each $\alpha \in \mathcal{F} \backslash \mathcal{H}$ using Theorem 2.1.1 and all corresponding β's are shown in Tables 1.1.1-1.1.3. This allows us to write the complete list of exceptional filling instructions in M_5 up to the action of (1.1)-(1.13). The set of isolated exceptional filling instructions in $\mathcal{F}_k$ is denoted by E_k and found in Tables 1.1.1-1.1.3, and $\bigcup E_k$ is denoted by E. The cardinalities of E and each E_k can be found in Table 2.1.1. $\square$

Proof of Proposition 1.1.2. Tables 1.1.4 and 1.1.5 are straightforward consequences of (1.3)-(1.13) and Table 1.1.1. $\square$

We will use Theorems 1.0.7 and 2.1.1 throughout the proof of Theorem 1.1.3. When two manifolds are seen to be equal using Theorem 1.0.7, we include the relevant map number in a subscript.

Proof of Theorem 1.1.3. Recall that for a hyperbolic manifold M a filling instruction α is exceptional if $M(\alpha)$ is non-hyperbolic, and it is isolated exceptional if α is exceptional and $M(\beta)$ is hyperbolic for all $\beta \subsetneq \alpha$. It immediately follows from this definition that if α is exceptional then there exists $\beta \subseteq \alpha$ that is exceptional. However, as we will now show, it is not always true that if α contains an isolated exceptional β then α is exceptional, but we will later prove that this is the case for $M = M_5$.

Myers shows in [**Mye**] that any 3-manifold is of the form $M(\beta)$ for some simple 3-manifold M and some slope β. By the Geometrization theorem, a simple manifold with boundary is hyperbolic. Now suppose that β is a slope on a boundary component of M and that $M(\beta) = X\#(S^3\backslash K)$ with hyperbolic X and $K \subset S^3$ a knot (such an M and β exist by [**Mye**]). If μ is a meridian of K then $M(\beta, \mu) = X$, so (β, μ) is not exceptional, but β is exceptional.

Let α be a filling instruction on M_5 containing an isolated exceptional filling instruction β. We will now show that α is exceptional. If $M_5(\beta)$ is closed then $\alpha = \beta$ and there is nothing to show. If $\partial M_5(\beta) \neq \varnothing$ then $M_5(\beta)$ is a graph manifold or $(P \times S^1)/\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ (see the proof of Theorems 1.1.1 and 3.0.1). Fillings of graph manifolds are graph manifolds and fillings of $(P \times S^1)/\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ are toroidal. So $M_5(\alpha)$ is not hyperbolic and our claim is proved.

Let α be an exceptional filling instruction containing an isolated exceptional filling instruction β. Theorem 1.1.1 implies that β either belongs to $[\![(1)]\!]$, $[\![(-1_0, -1_1)]\!]$, $[\![(-1, -3, -2, -2, -3)]\!]$, $[\![(-2, -\frac{1}{2}, 3, 3, -\frac{1}{2})]\!]$, or $[\![(-2, -2, -2, -2, -2)]\!]$, or that β factors through M_3.

The cases when β does not factor through M_3 constitute cases 1, 2, and 5 in the statement of Theorem 1.1.3. We will start the proof by treating these cases.

Case 1: If $\beta \in [\![(1_0)]\!]$ then Theorem 1.0.7 implies that $[\beta]$ is one of $[(1_0)]$, $[(0_0)]$, or $[(\infty_0)]$. In the proof of Theorem 1.1.1 we saw that

$$M_5\big(1, \tfrac{p}{q}, \tfrac{r}{s}, \tfrac{u}{v}, \tfrac{x}{y}\big) = F\big(\tfrac{p-q}{q}, \tfrac{r}{s}, \tfrac{u}{v}, \tfrac{x-y}{y}\big),$$

where $\frac{p}{q}, \dots, \frac{x}{y} \in \mathbb{Q} \cup \{\varnothing, \infty\}$. For $[\alpha] = [(0, \frac{p}{q}, \frac{r}{s}, \frac{u}{v}, \frac{x}{y})]$, and $[\alpha] = [(\infty, \frac{p}{q}, \frac{r}{s}, \frac{u}{v}, \frac{x}{y})]$ with $\frac{p}{q}, \dots, \frac{x}{y} \in \mathbb{Q} \cup \{\varnothing, \infty\}$ we use the maps (1.4) and (1.13) respectively and then Theorem 2.1.1 to get

$$M_5\big(0, \tfrac{p}{q}, \tfrac{r}{s}, \tfrac{u}{v}, \tfrac{x}{y}\big) \underset{(1.4)}{=} M_5\big(1, \tfrac{x-y}{x}, \tfrac{r-s}{r}, \tfrac{q}{q-p}, \tfrac{u}{v}\big) = F\big(- \tfrac{y}{x}, \tfrac{r-s}{r}, \tfrac{q}{q-p}, \tfrac{u-v}{v}\big),$$

and

$$M_5\big(\infty, \tfrac{p}{q}, \tfrac{r}{s}, \tfrac{u}{v}, \tfrac{x}{y}\big) \underset{(1.3)}{=} M_5\big(\tfrac{s}{r}, \tfrac{y-x}{y}, 1, \tfrac{q-p}{q}, \tfrac{v}{u}\big) = F\big(\tfrac{s}{r}, -\tfrac{x}{y}, -\tfrac{p}{q}, \tfrac{v}{u}\big),$$

which is what we wanted.

Case2: If β belongs to $[\![(-1_0,-1_1)]\!]$ then $\beta \in [((-1)_0,(-1)_1)] \cup [((\frac{1}{2})_0,2_1)] \cup [((\frac{1}{2})_0,(\frac{1}{2})_2)] \cup [((-1)_0,2_2)]$ by Theorem 1.0.7. By Theorem 2.1.1 and Theorem 1.0.7, we have:

$$M_5\big(-1,-1,\tfrac{p}{q},\tfrac{r}{s},\tfrac{u}{v}\big) = M_5\big(0,\tfrac{p}{q},-1,\tfrac{r-s}{s},\tfrac{u}{v}\big) \underset{(1.4)}{=} M_5\big(1,\tfrac{u-v}{u},2,\tfrac{q}{q-p},\tfrac{r-s}{s}\big) \\ = F\big(-\tfrac{v}{u},2,\tfrac{q}{q-p},\tfrac{r-2s}{s}\big),$$

$$M_5\big(\tfrac{1}{2},2,\tfrac{p}{q},\tfrac{r}{s},\tfrac{u}{v}\big) \underset{(1.3)}{=} M_5\big(\tfrac{q}{p},\tfrac{v-u}{v},-1,-1,\tfrac{s}{r}\big) = M_5\big(\tfrac{q-p}{p},-1,\tfrac{v-u}{v},0,\tfrac{s}{r}\big) \\ \underset{(1.13)}{=} M_5\big(\tfrac{q-2p}{q-p},\tfrac{-u}{v-u},1,-1,\tfrac{r}{r-s}\big) = F\big(\tfrac{q-2p}{q-p},\tfrac{v}{u-v},-2,\tfrac{r}{r-s}\big),$$

$$M_5\big(\tfrac{1}{2},\tfrac{p}{q},\tfrac{1}{2},\tfrac{r}{s},\tfrac{u}{v}\big) \underset{(1.13)}{=} M_5\big(-1,-1,\tfrac{s}{s-r},\tfrac{p}{q},\tfrac{v}{v-u}\big) = M_5\big(0,\tfrac{s}{s-r},-1,\tfrac{p-q}{q},\tfrac{v}{v-u}\big) \\ \underset{(1.4)}{=} M_5\big(1,-\tfrac{u}{v},2,\tfrac{r-s}{r},\tfrac{p-q}{q}\big) = F\big(\tfrac{u-v}{v},2,\tfrac{r-s}{r},\tfrac{p-2q}{q}\big),$$

$$M_5\big(-1,\tfrac{p}{q},2,\tfrac{r}{s},\tfrac{u}{v}\big) =_{(1.10)} M_5\big(\tfrac{r}{r-s},-1,-1,\tfrac{q}{p},\tfrac{v-u}{v}\big) = M_5\big(\tfrac{r}{r-s},0,\tfrac{q}{p},-1,-\tfrac{u}{v}\big) \\ \underset{(1.3)}{=} M_5\big(\tfrac{p}{q},\tfrac{v+u}{v},\tfrac{r}{s},1,-1\big) = F\big(\tfrac{p}{q},\tfrac{v+u}{v},\tfrac{r-s}{s},-2\big).$$

This completes case 2.

Case 5: For $\beta = \alpha$ in one of $[\![(-1,-3,-2,-2,-3)]\!]$, $[\![(-2,-\frac{1}{2},3,3,-\frac{1}{2})]\!]$, or $[\![(-2,-2,-2,-2,-2)]\!]$ Theorem 1.1.1 tells us that $M_5(\alpha)$ is homeomorphic to m_1, m_2, m_3 respectively. This completes Case 5.

Now we look at the cases where α factors through M_3, that is, when β belongs to $[\![(-1_0,-1_2)]\!]$ or $[\![(-1_0,-2_1)]\!]$.

Case 3: If β belongs to $[\![(-1_0,-1_2)]\!]$ then we have $\alpha \sim (-1,\frac{p}{q},-1,\frac{r}{s},\frac{u}{v})$ for some $\frac{p}{q},\frac{r}{s},\frac{u}{v} \in \mathbb{Q} \cup \{\varnothing,\infty\}$, and Theorem 2.1.1 implies that

$$M_5(\alpha) = M_3\big(\tfrac{p+2q}{q},\tfrac{r+s}{s},\tfrac{u+v}{v}\big),$$

which is what we wanted.

Case 4: If β belongs to $[\![(-1_0,-2_1)]\!]$ then we have $\alpha \sim (-1,-2,\frac{p}{q},\frac{r}{s},\frac{u}{v})$ for some $\frac{p}{q},\frac{r}{s},\frac{u}{v} \in \mathbb{Q} \cup \{\varnothing,\infty\}$, and Theorem 2.1.1 implies that

$$M_5(\alpha) = M_3\big(\tfrac{p+q}{q},\tfrac{r}{s},\tfrac{u+2v}{v}\big),$$

which completes Case 4.

This concludes the proof. $\square$

2.2. Exceptional fillings of M_5

To describe the exceptional fillings of M_5 we will now set out a number of standard identities (see [**FM**] for details). For a surface E with $\partial E \neq \varnothing$ we denote by E' the surface obtained by gluing a copy of D^2 to one of the boundary components of E. Details of the choice of bases on $\partial(E \times S^1)$ can be found in Section 1.2.

Lemma 2.2.1. *Let X and Y be 3-manifolds with fixed homology bases on ∂X and ∂Y, and E be a surface with boundary, G be a closed surface, and $p_i, q_i \in \mathbb{Z}$. Then the following identities hold:*

Seifert manifolds:

$$\big(G,(p_1,q_1),(p_2,q_2),\dots,(p_k,q_k)\big) = \big(G,(p_1,q_1-np_1),(p_2,q_2+np_2),\dots,(p_k,q_k)\big), \tag{2.2}$$

$$\big(E,(1,q_1),(p_2,q_2),\dots,(p_k,q_k)\big) = \big(E',(p_2,q_2+q_1p_2),\dots,(p_k,q_k)\big), \tag{2.3}$$

$$\big(E,(p_1,q_1),(p_2,q_2),\dots\big) = \big(E,(p_1,q_1-np_1),(p_2,q_2),\dots\big). \tag{2.4}$$

Small Seifert manifolds:

$$(S^2,(p,q)) = L(q,p), \tag{2.5}$$

$$(S^2,(p,q),(r,s)) = L(ps+rq,\, ps'+r'q) \text{ where } rs'-sr'=1, \tag{2.6}$$

$$\big(S^2,(0,1),(p,q),(r,s)\big) = L(p,q)\#L(r,s). \tag{2.7}$$

Graph Manifolds:

$$X \bigcup\nolimits_B Y = Y \bigcup\nolimits_{B^{-1}} X, \tag{2.8}$$

$$(D,(p,q)) \bigcup\nolimits_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} (E,\dots) = (E',(ap-bq,cp-dq),\dots), \tag{2.9}$$

$$\big(E,(p,q),\dots\big) \bigcup\nolimits_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} X = \big(E,(p,q+kp),\dots\big) \bigcup\nolimits_{\begin{pmatrix} a+kb & b \\ c+kd & d \end{pmatrix}} X, \tag{2.10}$$

$$\big(D^2,(0,1),(p,q)\big) \bigcup\nolimits_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \big(E,\dots\big) = L(p,q)\#\big(E',(b,d)\dots\big), \tag{2.11}$$

$$(E,(p_1,q_1),\dots,(p_k,q_k)) \bigcup_{\begin{pmatrix} a & c \\ b & d \end{pmatrix}} X = (E,(p_1,-q_1),\dots,(p_k,-q_k)) \bigcup_{\begin{pmatrix} -a & c \\ -b & d \end{pmatrix}} X. \tag{2.12}$$

Using Lemma 2.2.1 we can easily complete the proofs of Proposition 1.2.3 and Corollary 1.2.4.

Proof of Proposition 1.2.3. We recall that Corollary 1.2.3 gives a procedure to produce a manifold $F(\alpha)$ when given a filling instruction α for F, and we recall from Chapter 1 that we say that a pair (p_i, q_i) is *genuine* if $|p_i| \geq 2$.

Equations (1.14) and (1.15) describe the full action of the symmetry group of the link M4CL on the filling instructions on the exterior F; this can be seen directly from Figure 2.1.3 and verified with SnapPea. This confirms the first assertion in Proposition 1.2.3.

Now let α be a filling instruction for F. It is self-evident that either α contains only genuine slopes, or contains at least one slope of the form $\frac{0}{1}$, or contains no slope of the form $\frac{0}{1}$ and at least one slope of the form $\frac{1}{n}$. It remains only to verify the final identities on $F(\alpha)$ in Case 1, Case 2 and Case 3. To do this we use the observation from Example 2.1.2 and Figure 2.1.3 that

$$F(\tfrac{p}{q},\tfrac{r}{s},\tfrac{u}{v},\tfrac{x}{y}) = \big(D,(p,q),(u,v)\big) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \big(D,(r,s),(x,y)\big)$$

where $\frac{p}{q},\frac{r}{s},\frac{u}{v},\frac{x}{y} \in \mathbb{Q} \cup \{\infty, \varnothing\}$.

Case 1: If all the slopes in α are genuine then we need to show that for every filling instruction $\alpha = (\frac{p}{q},\frac{r}{s},\frac{u}{v},\frac{x}{y})$ on F, that the gluing torus between the two Seifert pieces of $X(\frac{p}{q},\frac{u}{v}) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} Y(\frac{r}{s},\frac{x}{y})$ is essential. This is equivalent to showing that $X(\frac{p}{q},\frac{u}{v})$ and $Y(\frac{r}{s},\frac{x}{y})$ are not homeomorphic to $D^2 \times S^1$ or $A \times S^1$ for $\frac{p}{q},\frac{u}{v},\frac{r}{s},\frac{x}{y}$ genuine, which follows from the classification of Seifert fibered spaces. This completes the proof of Case 1.

Case 2: When α contains a $\frac{0}{1}$ slope, using Identity 2.11 and the observation that $F(\alpha) = \big(D,(0,1),(a,b)\big) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \big(D,(c,d),(e,f)\big)$ for appropriate a,b,c,d,e,f, we see that $F(\alpha) = F(\alpha) = L(c,d) \# L(be+af, a'e+b'f)$ as claimed.

Case 3: In the case when α contains no $\frac{0}{1}$ slope but contains a non-genuine slope it is clear that α contains a slope of the form $\frac{1}{n}$ and, using Identities

1.14-1.15, that there exists $a,\dots f,n$ such that $\alpha = (\frac{1}{n},\frac{a}{b},\frac{-d}{c+nd},\frac{e}{f})$. Now

$$F(\alpha) = \big(D,(1,n),(-d,c+nd)\big)\bigcup_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}\big(D,(a,b),(e,f)\big),$$

and so, using Lemma 2.2.1, $F(\alpha) = X(\frac{a}{b},\frac{c}{d},\frac{e}{f}) = \big(S^2,(a,b),(c,d),(e,f)\big)\big)$. Identities 1.16 and 1.17 are now clear, and Cases 3(i), 3(ii), 3(iii) are a simple enumeration of all possibilities on the slopes $\frac{a}{b},\frac{c}{d},\frac{e}{f}$. The final identities follow from Lemma 2.2.1. □

Proof of Corollary 1.2.4. We recall that Corollary 1.2.4 includes a list of manifolds which are claimed to be all exceptional fillings of M_5, and that, with the exception of 5 manifolds, these manifolds are obtained by a Dehn filling along a slope $\frac{p}{q}$ of some hyperbolic $M_5(\alpha)$.

Let $\mathcal{L}_1$ denote the set of manifolds listed in the statement of Corollary 1.2.4 and $\mathcal{L}_2$ denote the set of exceptional fillings of M_5. By Theorem 1.1.3, every element of $\mathcal{L}_2$ is equal to a filling of F or to an exceptional filling of M_3, or it is in $\{m_1,m_2,m_3\}$. Using Proposition 1.2.3 and Theorem 3.0.1 one clearly sees that $\mathcal{L}_2 \subset \mathcal{L}_1$.

The second assertion of Theorem 1.2.4 obviously implies the first one, subject to there existing an exceptional filling instruction α on $M_5(\alpha)$ with $M_5(\alpha) = m$ for each $m \in \Big\{T/_{\begin{pmatrix}n&1\\-1&0\end{pmatrix}} : n=-1,0,1,2,3\Big\}$. Using Theorems 2.1.1 and 3.0.1 it is easy to see that these m's are realised as $M_5(\alpha)$ for $\alpha = (-1,-2,n-1,-1,-3)$, $n = \{-1,0,1,2,3\}$.

We proceed by proving the second statement directly by explicitly exhibiting $(\alpha,\frac{p}{q})$, where α is a hyperbolic filling instruction on M_5, with $M_5(\alpha)(\frac{p}{q}) = m$ for every $m \in \mathcal{L}_1$, thus showing that $\mathcal{L}_1 \subset \mathcal{L}_2$ and completing the proof.

Toroidal manifolds: For every choice of genuine or blank $\frac{p}{q},\frac{r}{s},\frac{u}{v},\frac{x}{y} \in \mathbb{Q}\cup\{\varnothing\}$ choose a corresponding $\alpha = \big(\frac{p}{q+kp}+1,\frac{r}{s+nr},\frac{u}{v-ku},\frac{x}{y-nx}+1,\varnothing\big)$, where a slope is $\varnothing$ if the corresponding original slope was $\varnothing$, and k and n are large enough to make each slope of α have length greater that 6 with respect to some horospherical cusp section. By Theorem 2.1.4 this ensures that α is a hyperbolic filling instruction.

Recall from Lemma 1.2.3 that

$$M_5(\alpha)(1) = F\big(\tfrac{p}{q+kp},\tfrac{r}{s+nr},\tfrac{u}{v-ku},\tfrac{x}{y-nx}\big) = X\bigcup_{\begin{pmatrix}0&1\\1&0\end{pmatrix}} Y,$$

where

$$X = \begin{cases} P \times S^1 & \text{if } \frac{p}{q+kp}, \frac{u}{v-ku} = \varnothing \\ \big(A, (\scriptstyle p,\, q+kp \textstyle)\big) & \text{if } \frac{u}{v-ku} = \varnothing \\ \big(A, (\scriptstyle u,\, v-ku \textstyle)\big) & \text{if } \frac{p}{q+kp} = \varnothing \\ \big(D, (\scriptstyle p,\, q+kp \textstyle), (\scriptstyle u,\, v-ku \textstyle)\big) & \text{otherwise} \end{cases}$$

and

$$Y = \begin{cases} P \times S^1 & \text{if } \frac{r}{s+nr}, \frac{x}{y-nx} = \varnothing \\ \big(A, (\scriptstyle r,\, s+nr \textstyle)\big) & \text{if } \frac{x}{y-nx} = \varnothing \\ \big(A, (\scriptstyle x,\, y-nx \textstyle)\big) & \text{if } \frac{r}{s+nr} = \varnothing \\ \big(D, (\scriptstyle r,\, s+nr \textstyle), (\scriptstyle x,\, y-nx \textstyle)\big) & \text{otherwise} \end{cases}$$

Identities (2.2) and (2.4) from Lemma 2.2.1 imply that

$$\big(E, (\scriptstyle \alpha,\, \beta \textstyle), (\scriptstyle \gamma,\, \delta \textstyle)\big) \bigcup_{\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)} Y = \big(E, (\scriptstyle \alpha,\, \beta+\alpha \textstyle), (\scriptstyle \gamma,\, \delta-\gamma \textstyle)\big) \bigcup_{\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)} Y,$$

and that

$$\big(E, (\scriptstyle \alpha,\, \beta \textstyle)\big) \bigcup_{\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)} Y \cong \big(E, (\scriptstyle \alpha,\, \beta+\alpha \textstyle)\big) \bigcup_{\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)} Y$$

when ∂E has more than one boundary component, from which one sees that the constructed $\{M_5(\alpha)(1)\}$ contains all of

$$X \bigcup_{\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)} Y \text{ for } X, Y \in \{P \times S^1, \big(A, (\scriptstyle p,\, q \textstyle)\big), \big(D, (\scriptstyle p,\, q \textstyle), (\scriptstyle r,\, s \textstyle)\big) : p, q, r, s \in \mathbb{Q}, |p|, |r| > 1\}.$$

The manifolds

$$\big(D, (\scriptstyle 2,\, 1 \textstyle), (\scriptstyle 2,\, 1 \textstyle)\big) \bigcup_B \big(D, (\scriptstyle 2,\, 1 \textstyle), (\scriptstyle 3,\, 1 \textstyle)\big) \text{ where } B \in \left\{ \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 5 \\ 1 & 4 \end{pmatrix} \right\},$$

$$\big(A, (\scriptstyle 2,\, -1 \textstyle)\big) \Big/ \left(\begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix}\right)$$

are realised by the following; $M_5(-5,-1,-2,-4)(-1)$, $M_5(-4,-1,-2,-3)(-2)$, $M_5(-3,-2,-2,-3)(-1)$, $M_5(-2,-2,-2,-2)(-2)$, and $M_5(-\frac{1}{2}, 3, 3, -\frac{1}{2})(-2)$ respectively. Using Tables 1.1.4-1.1.5 it is easy to check that α is a hyperbolic filling instruction for each of these $M_5(\alpha)(\beta)$.

The remaining toroidal fillings of M_5 are fillings of $(P \times S^1) \big/ \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Let $\alpha = (-1, -2, \frac{r}{s} - 1, -1, \varnothing)$ be a filling instruction on M_5, and $\beta = (\frac{r}{s}, -1)$ be a filling instruction on M_3 (where $\frac{r}{s} = \varnothing$ if the $\frac{r}{s} - 1$ slope on M_5 is $\varnothing$). From [**MP**] (see Chapter 3) we have

$$M_5(\alpha)(-3) = M_3(\beta)(-1) = \begin{cases} T/\begin{pmatrix} n & 1 \\ -1 & 0 \end{pmatrix} & \text{if } \frac{r}{s} = n \\ \big(A, (r, s)\big)/\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{if } \frac{r}{s} \notin \mathbb{Z} \\ (P \times S^1)/\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{if } \frac{r}{s} - 1 = \varnothing. \end{cases}$$

Moreover $\alpha = (-1, -2, \frac{r}{s} - 1, -1, \varnothing)$ is hyperbolic provided $\frac{r}{s} \notin \{-1, 0, 1, 2, 3\}$ (see Theorem 3.0.1).

This shows that all toroidal manifolds in $\mathcal{L}_1$ are contained in $\mathcal{L}_2$, and that, with the exceptions of $\left\{T/\begin{pmatrix} n & 1 \\ -1 & 0 \end{pmatrix} : n = -1, 0, 1, 2, 3\right\}$, every toroidal manifold in $\mathcal{L}_2$ is obtained as an exceptional Dehn filling of some hyperbolic $M_5(\alpha)$.

Reducible manifolds: Given genuine slopes $\frac{p}{q}$, $\frac{r}{s}$ we choose integers k, n large enough that $\alpha = (\frac{1}{k} + 1, \frac{p}{q-np}, -\frac{1}{k}, \frac{r}{s+nr} + 1)$, $\beta = (1 + \frac{1}{k}, \frac{p}{q+np}, -\frac{1}{k}, \varnothing)$, $\gamma = (1 + \frac{1}{k}, \varnothing, -\frac{1}{k}, \varnothing)$ are all hyperbolic filling instructions on M_5, and we use Proposition 1.2.3 to see that

$$M_5(\alpha)(1) \underset{(2.1.1)}{=} F\big(\tfrac{1}{k}, \tfrac{p}{q-np}, -\tfrac{1}{k}, \tfrac{r}{s+nr}\big) = L(p, q)\#L(r, s)$$

$$M_5(\beta)(1) \underset{(2.1.1)}{=} F\big(\tfrac{1}{k}, \tfrac{p}{q+np}, -\tfrac{1}{k}, \varnothing\big) = D \times S^1\#L(p, q)$$

$$M_5(\gamma)(1) \underset{(2.1.1)}{=} F\big(\tfrac{1}{k}, \varnothing, -\tfrac{1}{k}, \varnothing\big) = D \times S^1\#D \times S^1.$$

This shows that all reducible manifolds in $\mathcal{L}_1$ are contained in $\mathcal{L}_2$, and that every reducible manifold in $\mathcal{L}_2$ is obtained by an exceptional Dehn filling along a slope on some boundary component of $M_5(\alpha)$.

Small Seifert manifolds: Given genuine slopes $\frac{p}{q}$, $\frac{r}{s}$, $\frac{u}{v}$ we choose integers k, n large enough that $\alpha = \big(1 + \frac{1}{k}, \frac{p}{q+np}, \varnothing, \frac{u}{v-nu} + 1\big)$, $\beta = \big(1 + \frac{1}{k}, \frac{1}{n}, \varnothing, 1 - \frac{1}{n}\big)$, $\gamma = \big(1 + \frac{1}{k}, \frac{p}{q+np}, \varnothing, \varnothing\big)$, $\delta = \big(\varnothing, \frac{1}{k}, \varnothing, \varnothing\big)$ are non-exceptional filling instructions of M_5, and we use Lemma 2.2.1 to see that

$$M_5(\alpha)(1) = F\big(\tfrac{1}{k}, \tfrac{p}{q+np}, \varnothing, \tfrac{u}{v-nu}\big) = \big(D, (p, q), (u, v)\big)$$

$$M_5(\beta)(1) = F\big(\tfrac{1}{k}, \tfrac{1}{n}, \varnothing, -\tfrac{1}{n}\big) = D \times S^1$$

$$M_5(\gamma)(1) = F\big(\tfrac{1}{k}, \tfrac{p}{q+np}, \varnothing, \varnothing\big) = \big(A, (p, q)\big)$$

$$M_5(\delta)(1) = F\big(\varnothing, \tfrac{1}{k}, \varnothing, \varnothing\big) = P \times S^1.$$

This shows that all small Seifert spaces with boundary are in $\mathcal{L}_2$. We now turn to the case of closed small Seifert spaces.

Given genuine slopes $\frac{p}{q}, \frac{r}{s}, \frac{u}{v}$ we take integers k, n large enough that $\alpha = \big(1 - \frac{q}{p+kq}, \frac{r}{s}, \frac{1}{k}, \frac{u}{v} + 1\big)$, $\beta = \big(1 + \frac{1}{k}, \frac{-q}{qn+p}, -\frac{1}{k}, \frac{1}{n} + 1\big)$, $\gamma = \big(1 + \frac{1}{k-n}, \frac{k}{1-kn}, \frac{1}{n}, \frac{1}{m} + 1\big)$, $\delta = \big(3, \frac{p}{q-np}, 2, \frac{1+n}{n}\big)$, $\epsilon = \big(3, -\frac{1}{k}, 2, \frac{1}{k}\big)$ are hyperbolic filling instructions of M_5, and use Theorem 1.1.3 to see that

$$M_5(\alpha)(1) = \big(S^2, {\scriptstyle(p,\,q)}, {\scriptstyle(r,\,s)}, {\scriptstyle(u,\,v)}\big), \quad M_5(\beta)(1) = L(q,p), \quad M_5(\gamma)(1) = S^2 \times S^1$$

$$M_5(\delta)(1) = \big(\mathbb{RP}^2, {\scriptstyle(p,\,q)}\big), \qquad M_5(\epsilon)(1) = \mathbb{RP}^2 \times S^1.$$

This shows that all small Seifert spaces are contained in $\mathcal{L}_2$, and that every small Seifert space in $\mathcal{L}_2$ is obtained as an exceptional Dehn filling along of some $M_5(\alpha)$. This completes the final case of Corollary 1.2.4 and completes the proof that $\mathcal{L}_1 = \mathcal{L}_2$. $\square$

Remark In the construction of the $(\alpha, \frac{p}{q})$ with $M_5(\alpha)(\frac{p}{q}) = m$ in the proof of Theorem 1.2.4, infinitely many inequivalent α's are given for many m's.

2.3. Exceptional slopes on $M_5(\alpha)$

In this section we prove Corollaries 1.3.1 and 1.3.2.

Proof of Corollary 1.3.1. Theorem 1.1.1 implies that all slopes in $[\![(1)]\!]$ are exceptional. By Theorem 1.0.7

$$[\![(1)]\!] = [(1)] \sqcup [(\infty)] \sqcup [(0)]$$

so $e(M_5(\alpha)) \geq 3$ for every hyperbolic filling instruction α containing at least one $\varnothing$. We will now describe all such α's not factoring through M_4 with $e(M_5(\alpha)) > 3$. Let $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \varnothing)$ be such an α, so $\alpha_i \notin \{-1, \frac{1}{2}, 2\}$. We define (α, β) to be $(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta)$. If β is an exceptional slope of $M_5(\alpha)$ then (α, β) contains an isolated exceptional filling instruction by Theorem 1.1.3. Any such isolated exceptional filling instruction contains at most one slope in $[\![(-1)]\!]$, and contains two slopes in $[\![(-2)]\!]$ (see Tables 1.1.1-1.1.3). Thus at least one of the slopes in α belongs to $[\![(-2)]\!]$. It is a routine consequence of Theorem 1.0.7 that we may assume without loss of generality that $\alpha_0 = -2$. Moreover, if (α, β) contains an isolated exceptional filling instruction then either β is a slope in $[\![(1)]\!]$, or β is a slope in $[\![(-1)]\!]$ and (α,β) factors through M_3, or $M_5(\alpha,\beta) \in \{m_1, m_2, m_3\}$ (see Tables 1.1.1-1.1.3). If (α,β) factors through M_3 then $\beta \in \{-1, \frac{1}{2}, 2\}$.

We define the following sets of (α,β)'s;

- We define l to be the set of exceptional (α,β)'s such that $M_5(\alpha,\beta)$ is in $\{m_1, m_2, m_3\}$;
- We define l_{-1} to be the set of exceptional (α,β)'s factoring through M_3 with $\beta = -1$;
- We define $l_{\frac{1}{2}}$ to be the set of exceptional (α,β)'s factoring through M_3 with $\beta = \frac{1}{2}$;
- We define l_2 to be the set of exceptional (α,β)'s factoring through M_3 with $\beta = 2$;

We know that, up to $\sim$, every α with $M_5(\alpha)$ a hyperbolic manifold, α not factoring through M_4, and $e(M_5(\alpha)) > 3$ has $\alpha_0 = -2$. The set $l_{-1} \cup l_{\frac{1}{2}} \cup l_2$ is the set of all exceptional filling instructions (α,β) factoring through M_3 with $M_5(\alpha)$ hyperbolic not factoring through M_4 and $\alpha_0 = -2$, and the set l is the set of all exceptional filling instructions (α,β) not factoring through M_3 with $\alpha_0 = -2$.

So, every hyperbolic filling instruction γ not factoring through M_4 with $e(M_5(\gamma)) > 3$ is equivalent to a filling instruction of the form $(-2, \frac{p}{q}, \frac{r}{s}, \frac{u}{v})$, every $(-2, \frac{p}{q}, \frac{r}{s}, \frac{u}{v})$ with $e(M_5(-2, \frac{p}{q}, \frac{r}{s}, \frac{u}{v})) > 3$ and $(-2, \frac{p}{q}, \frac{r}{s}, \frac{u}{v})$ not factoring through M_4 is some α in $l \cup l_{-1} \cup l_{\frac{1}{2}} \cup l_2$, and every exceptional slope on $M(-2, \frac{p}{q}, \frac{r}{s}, \frac{u}{v})$ not in $\{0, 1, \infty\}$ is some β in $l \cup l_{-1} \cup l_{\frac{1}{2}} \cup l_2$. Now we can easily describe, up to $\sim$, all filling instructions not factoring through M_4 with more than 3 exceptional slopes.

Let $p : l \cup l_{-1} \cup l_{\frac{1}{2}} \cup l_2 \to A$ be defined by $(\alpha,\beta) \mapsto \alpha$. For α in $p(l \cup l_{-1} \cup l_{\frac{1}{2}} \cup l_2)$ define B_α to be the set of all β's such that (α,β) is contained in $l \cup l_{-1} \cup l_{\frac{1}{2}} \cup l_2$. It is clear that $E\big(M_5(\alpha)\big) = \{0, 1, \infty\} \cup B_\alpha$ and that $\{\big(M_5(\alpha), E(M_5(\alpha))\big)\}_\alpha$ is a complete list of all $\big(M_5(\alpha), E(M_5(\alpha))\big)$ pairs with $\alpha = \big(-2, \frac{p}{q}, \frac{r}{s}, \frac{u}{v}, \varnothing\big)$, $M_5(\alpha)$ hyperbolic and $e\big(M_5(\alpha)\big) > 3$. We now explicitly construct the sets $l, l_{-1}, l_{\frac{1}{2}}, l_2$.

Construction of the set l: We can see using Theorem 1.0.7 that every (α,β) with $M_5(\alpha,\beta) = m_i$ with $\alpha_0 = -2$ and α hyperbolic not factoring through M_4 is contained in the following set

$$l = \Big\{\big(-2, \tfrac{1}{4}, \tfrac{3}{2}, \tfrac{4}{3}, \tfrac{1}{2}\big), \big(-2, -\tfrac{1}{2}, 3, 3, -\tfrac{1}{2}\big), \big(-2, \tfrac{1}{3}, 3, \tfrac{1}{3}, -2\big), \big(-2, -2, \tfrac{1}{3}, 3, \tfrac{1}{3}\big), \\ \big(-2, -\tfrac{1}{2}, -2, \tfrac{3}{2}, \tfrac{3}{2}\big), \big(-2, \tfrac{3}{2}, \tfrac{3}{2}, -2, -\tfrac{1}{2}\big), \big(-2, -2, -2, -2, -2\big), \big(-2, \tfrac{1}{3}, \tfrac{3}{2}, \tfrac{3}{2}, \tfrac{1}{3}\big)\Big\}.$$

Before constructing the sets $l_{-1}, l_{\frac{1}{2}}, l_2$ we make some initial remarks. If (α,β) factors through M_3 then (α,β) contains a filling instruction in $[\![(-1_0, -2_1)]\!]$. By

Theorem 1.0.7 we have

$$[\![((-1)_0,(-2)_1)]\!] = \Big\{[((\tfrac{1}{2})_0,(\tfrac{2}{3})_2)],[((\tfrac{1}{2})_0,3_1)],[((\tfrac{1}{2})_0,(\tfrac{3}{2})_1)],[((\tfrac{2}{3})_0,2_1)],[((\tfrac{1}{2})_0,(\tfrac{1}{3})_2)],\\ [(2_0,(-\tfrac{1}{2})_2)],[((\tfrac{1}{3})_0,2_1)],[((-1)_0,3_2)],[((-1)_0,(-2)_1)],[((-1)_0,(\tfrac{3}{2})_2)],[(2_0,(-2)_2)],\\ [((-1)_0,(-\tfrac{1}{2})_1)]\Big\}$$

If (α,β) contains one of the elements of $[\![((-1)_0,(-2)_1)]\!]$ then (α,β) factors through M_3 and β is an element of $\{-1,\frac{1}{2},2\}$. We will see that the requirements of α being hyperbolic and not factoring through M_4 allow us to completely construct $l_{-1},l_{\frac{1}{2}},l_2$.

Construction of the set l_{-1}: In this case, for $\alpha=(-2,\frac{p}{q},\frac{r}{s},\frac{u}{v})$, we have $M_5(\alpha)(-1)=M_3(\frac{p+q}{q},\frac{r}{s},\frac{u+2}{v})$, and (α,β) is an exceptional filling of M_5 if and only if $(\frac{p+q}{q},\frac{r}{s},\frac{u+2v}{v})$ contains an isolated exceptional filling instruction on M_3 (see Theorem 3.0.1). For α non-exceptional and not factoring through M_4, no slope in α is in $[(1)]$ or $[(-1)]$; that is $\frac{p+q}{q}\notin\{\infty,2,1,0,\frac{3}{2},3\}$, $\frac{r}{s}\notin\{\infty,1,0,-1,\frac{1}{2},2\}$, $\frac{u+2v}{v}\notin\{\infty,3,2,1,\frac{5}{2},4\}$. With these conditions and Theorem 3.0.1 it is easy to see that β is an exceptional slope on a hyperbolic $M_5(\alpha)$ if and only if one of the following holds:

- $\frac{r}{s}=3$
- $\frac{u}{v}+2=0$
- $(\frac{p}{q}+1,\frac{u}{v}+2)$ belongs to $\{(-1,-1),(\frac{5}{2},\frac{3}{2}),(4,\frac{1}{2})\}$
- $(\frac{r}{s},\frac{p}{q}+1)$ belongs to $\{(\frac{3}{2},\frac{5}{2}),(4,\frac{1}{2})\}$
- $(\frac{r}{s},\frac{u}{v}+2)$ belongs to $\{(\frac{5}{2},\frac{3}{2}),(4,\frac{1}{2})\}$
- $(\frac{p}{q}+1,\frac{r}{s},\frac{u}{v}+2)$ belongs to

$$\Big\{(5,5,\tfrac{1}{2}),(\tfrac{1}{2},5,5),(4,4,\tfrac{2}{3}),(4,\tfrac{3}{2},\tfrac{3}{2}),(4,\tfrac{1}{3},-1),(\tfrac{1}{3},4,-1),(-1,4,\tfrac{1}{3}),(\tfrac{8}{3},\tfrac{3}{2},\tfrac{3}{2}),\\ (\tfrac{5}{2},\tfrac{5}{2},\tfrac{4}{3}),(\tfrac{5}{2},\tfrac{5}{3},\tfrac{5}{3}),(\tfrac{5}{3},\tfrac{5}{2},\tfrac{5}{3}),(\tfrac{7}{3},\tfrac{7}{3},\tfrac{3}{2}),(\tfrac{7}{3},\tfrac{3}{2},\tfrac{7}{3}),(-1,-2,-2),(-2,-2,-1),\\ (-1,-2,-3),(-3,-2,-1),(-2,-3,-1),(-1,-3,-2),(-1,-2,-4),(-4,-2,-1),\\ (-1,-4,-2),(-2,-4,-1),(-1,-2,-5),(-5,-2,-1),(-1,-5,-2),(-2,-5,-1),\\ (-1,-3,-3),(-3,-3,-1),(-2,-2,-2))\Big\}.$$

Thus, the set of all $(-2, \frac{p}{q}, \frac{r}{s}, \frac{u}{v}, -1)$ constructed in the above analysis is the set l_{-1}.

Construction of the set $l_{\frac{1}{2}}$: Reasoning as in the case $\beta = 2$ we see (α, β) factors through M_3 when one of $\frac{p}{q}, \frac{r}{s}$ is in $\{\frac{1}{3}, \frac{2}{3}\}$ or $\frac{u}{v} \in \{3, \frac{3}{2}\}$. We examine each of these 6 cases individually and enumerate all $(-2, \frac{p}{q}, \frac{r}{s}, \frac{u}{v}, \frac{1}{2})$ with the desired properties.

If $\frac{u}{v} = 3$ then

$$M_5\big(-2, \tfrac{p}{q}, \tfrac{r}{s}, 3, \tfrac{1}{2}\big) \underset{(1.5)}{=} M_5\big(-1, 3, \tfrac{s}{r}, \tfrac{q}{p}, -2\big) = M_3\big(5, \tfrac{s}{r}, \tfrac{q+p}{p}\big).$$

The result is that $\frac{1}{2}$ is an exceptional slope on $M_5\big(-2, \frac{p}{q}, \frac{r}{s}, \frac{u}{v}\big)$ if and only if one of the following holds; $\frac{s}{r} = 3$, $\big(\frac{s}{r}, \frac{q+p}{p}\big) = \big(\frac{3}{2}, \frac{5}{2}\big)$, $(4, \frac{1}{2})$, $(5, \frac{1}{2})$.

If $\frac{u}{v} = \frac{3}{2}$ then

$$M_5\big(-2, \tfrac{p}{q}, \tfrac{r}{s}, \tfrac{3}{2}, \tfrac{1}{2}\big) \underset{(1.9)}{=} M_5\big(-2, -2, \tfrac{s}{s-r}, \tfrac{p-q}{p}, -1\big) = M_3\big(-1, \tfrac{s}{s-r}, \tfrac{3p-q}{p}\big).$$

We conclude that $\frac{1}{2}$ is an exceptional slope on $M_5\big(-2, \frac{p}{q}, \frac{r}{s}, \frac{u}{v}\big)$ if and only if one of the following holds; $\frac{s}{s-r} = 3$, $\frac{3p-q}{p} = 0, -1$, $(\frac{s}{s-r}, \frac{3p-q}{p}\big) = (\frac{5}{2}, \frac{3}{2})$, $(4, \frac{1}{2})$, $(-2, -2)$, $(-2, -3)$, $(-3, -2)$, $(-2, -4)$, $(-2, -5)$, $(-5, -2)$, $(4, \frac{1}{3})$.

If $\frac{p}{q} = \frac{1}{3}$ then

$$M_5(\alpha)(\beta) \underset{(1.9)}{=} M_5\big(\tfrac{v}{v-u}, -2, \tfrac{s}{s-r}, -2, -1\big) = M_3\big(\tfrac{3v-2u}{v-u} - 2, \tfrac{2s-r}{s-r}\big).$$

The result is that $\frac{1}{2}$ is an exceptional slope on $M_5\big(-2, \frac{p}{q}, \frac{r}{s}, \frac{u}{v}\big)$ if and only if one of the following holds; $\frac{2s-r}{s-r} = 3$, $\frac{3v-2u}{v-u} = 0$, $(\frac{v}{v-u}, -2, \frac{s}{s-r}) = (-1, -1)$, $(\frac{v}{v-u}, -2, \frac{s}{s-r}) = (\frac{1}{2}, 4)$, $(\frac{v}{v-u}, -2, \frac{s}{s-r}) = (\frac{3}{2}, \frac{5}{2})$.

If $\frac{r}{s} = \frac{1}{3}$ then

$$M_5\big(-2, \tfrac{p}{q}, \tfrac{1}{3}, \tfrac{u}{v}, \tfrac{1}{2}\big) \underset{(1.4)}{=} M_5\big(\tfrac{1}{3}, -1, -2, \tfrac{q}{q-p}, \tfrac{u}{v}\big) = M_3\big(\tfrac{7}{3}, \tfrac{2q-p}{q-p}, \tfrac{u}{v}\big).$$

We see that $\frac{1}{2}$ is an exceptional slope on $M_5\big(-2, \frac{p}{q}, \frac{r}{s}, \frac{u}{v}\big)$ if and only if one of the following holds; $\frac{u}{v} = 3$, $(\frac{u}{v}, \frac{2q-p}{q-p})=(\frac{3}{2}, \frac{5}{2})$, $(\frac{3}{2}, \frac{7}{3})$.

If $\frac{p}{q} = \frac{2}{3}$ then

$$M_5(\alpha)(\beta) \underset{(1.7)}{=} M_5\big(\tfrac{2}{3}, \tfrac{r}{r-s}, -1, -2, \tfrac{u}{u-v}\big) = M_3\big(\tfrac{2}{3}, \tfrac{3r-2s}{r-s}, \tfrac{2u-v}{u-v}\big).$$

We find that $\frac{1}{2}$ is an exceptional slope on $M_5\big(-2, \frac{p}{q}, \frac{r}{s}, \frac{u}{v}\big)$ if and only if one of the following holds; $\frac{3r-2s}{r-s} = 0$, $(\frac{3r-2s}{r-s}, \frac{2u-v}{u-v}) = (-1, -1)$, $(\frac{3}{2}, \frac{5}{2})$.

If $\frac{r}{s} = \frac{2}{3}$ then

$$M_5(\alpha)(\beta) \underset{(1.7)}{=} M_5\big(\tfrac{2}{3}, -2, -1, \tfrac{p}{p-q}, \tfrac{u}{u-v}\big) = M_3\big(\tfrac{5}{3}, \tfrac{3p-2q}{p-q}, \tfrac{u}{u-v}\big).$$

The result is that $\frac{1}{2}$ is an exceptional slope on $M_5\big(-2, \frac{p}{q}, \frac{r}{s}, \frac{u}{v}\big)$ if and only if one of the following holds; $\frac{u}{u-v} = 3$, $(\frac{3p-2q}{p-q}, \frac{u}{u-v}) = (\frac{3}{2}, \frac{5}{2})$, $(\frac{1}{2}, 4)$, $(\frac{5}{3}, \frac{5}{2})$.

Thus, the set of all $(-2, \frac{p}{q}, \frac{r}{s}, \frac{u}{v}, \frac{1}{2})$ constructed in the above analysis is the set $l_{\frac{1}{2}}$.

Construction of the set l_2: For $(\alpha, \beta) = (-2, \frac{p}{q}, \frac{r}{s}, \frac{u}{v}, 2)$ to factor through M_3 we need one of $\frac{p}{q}, \frac{r}{s}$ to be in $\{-2, -\frac{1}{2}\}$ or $\frac{u}{v} \in \{\frac{1}{3}, \frac{2}{3}\}$ (see $[\![(-1_0, -2_1)]\!]$ shown above). We construct all (α, β) for each of these 6 cases individually.

If $\frac{p}{q} = -2$ then

$$M_5(\alpha)(2) =_{(1.13)} M_5\big(\tfrac{3}{2}, \tfrac{r-s}{r}, \tfrac{v}{v-u}, -2, -1\big) = M_3\big(\tfrac{7}{2}, \tfrac{r-s}{r}, \tfrac{2v-u}{v-u}\big).$$

When no $\frac{p}{q}, \frac{r}{s}, \frac{u}{v}$ is a slope in $[\![(1)]\!], [\![(-1)]\!]$ we have $\frac{r-s}{r} \notin \{\infty, 0, 1, 2, \frac{1}{2}, -1\}$ and $\frac{2v-u}{v-u} \notin \{2, \infty, 1, \frac{3}{2}, 0, 3\}$. From Theorem 1.0.7 we see that 2 is exceptional if and only if one of the following holds; $\frac{r-s}{r} = 3$, $(\frac{r-s}{r}, \frac{2v-u}{v-u}) = (\frac{3}{2}, \frac{5}{2})$, $(\frac{r-s}{r}, \frac{2v-u}{v-u}) = (4, \frac{1}{2})$.

If $\frac{r}{s} = -2$ then

$$M_5(\alpha)(\beta) \underset{(1.6)}{=} M_5\big(-1, -2, \tfrac{1}{3}, \tfrac{p-q}{p}, \tfrac{u-v}{u}\big) = M_3\big(\tfrac{4}{3}, \tfrac{p-q}{p}, \tfrac{3u-v}{u}\big).$$

Using Theorem 1.0.7 we get that 2 is exceptional on $M_5(\alpha)$ if and only if one of the following holds; $\frac{p-q}{p} = 3$, $\frac{3u-v}{u} = 0$, $(\frac{p-q}{p}, \frac{3u-v}{u}) = (4, \frac{1}{2})$, $(\frac{p-q}{p}, \frac{3u-v}{u}) = (\frac{5}{2}, \frac{3}{2})$.

If $\frac{p}{q} = -\frac{1}{2}$ then

$$M_5(\alpha)(\beta) \underset{(1.10)}{=} M_5\big(\tfrac{u}{u-v}, \tfrac{s-r}{s}, -\tfrac{1}{2}, -2, -1\big) = M_3\big(\tfrac{3u-2v}{u-v}, \tfrac{s-r}{s}, \tfrac{1}{2}\big).$$

We find that 2 is exceptional on $M_5(\alpha)$ if and only if one of the following holds; $\frac{s-r}{s} = 3$, $\frac{3u-2v}{u-v} = 0$, $(\frac{s-r}{s}, \frac{3u-2v}{u-v}) = (4, \frac{1}{2})$, $(\frac{s-r}{s}, \frac{3u-2v}{u-v}) = (\frac{5}{2}, \frac{3}{2})$.

If $\frac{r}{s} = -\frac{1}{2}$ then

$$M_5(\alpha)(\beta) \underset{(1.3)}{=} M_5\big(-2, -1, \tfrac{2}{3}, \tfrac{q-p}{q}, \tfrac{v}{u}\big) = M_3\big(\tfrac{8}{3}, \tfrac{q-p}{q}, \tfrac{v+u}{u}\big).$$

We observe that 2 is exceptional on $M_5(\alpha)$ if and only if one of the following holds; $\frac{q-p}{q} = 3$, $(\frac{q-p}{q}, \frac{v+u}{u}) = (4, \frac{1}{2})$, $(\frac{q-p}{q}, \frac{v+u}{u}) = (\frac{3}{2}, \frac{5}{2})$.

If $\frac{u}{v} = \frac{2}{3}$ then

$$M_5(\alpha)(\beta) \underset{(1.10)}{=} M_5\big(-2, \tfrac{s-r}{s}, -\tfrac{1}{2}, \tfrac{q}{p}, -1\big) = M_3\big(\tfrac{2s-r}{s}, -\tfrac{1}{2}, \tfrac{2p+q}{p}\big).$$

We get that 2 is exceptional on $M_5(\alpha)$ if and only if one of the following holds; $\frac{2p+q}{p} = 0$, $(\frac{s-r}{s}, \frac{q}{p}) = (-1,-1)$, $(\frac{2s-r}{s}, \frac{2p+q}{p}) = (4, \frac{1}{2})$, $(\frac{2s-r}{s}, \frac{2q+p}{p}) = (\frac{5}{2}, \frac{3}{2})$.

If $\frac{u}{v} = \frac{1}{3}$ then

$$M_5(\alpha)(\beta) \underset{(1.6)}{=} M_5\big(-1, \tfrac{r}{s}, \tfrac{1}{3}, \tfrac{p-q}{p}, -2\big) = M_3\big(\tfrac{r+2s}{s}, \tfrac{1}{3}, \tfrac{2p-q}{p}\big).$$

The result is that 2 is exceptional on $M_5(\alpha)$ if and only if one of the following holds; $\frac{r+2s}{s} = 0$, $(\frac{r+2s}{s}, \frac{2p-q}{p}) = (\frac{1}{2}, 4)$, $(\frac{r+2s}{s}, \frac{2p-q}{p}) = (\frac{3}{2}, \frac{5}{2})$, $(\frac{r+2s}{s}, \frac{2p-q}{p}) = (-1,-1)$, $(\frac{r+2s}{s}, \frac{2p-q}{p}) = (-1, 4)$.

Thus, the set of all $(-2, \frac{p}{q}, \frac{r}{s}, \frac{u}{v}, 2)$ constructed in the above analysis is the set l_2.

This completes the construction of $l \cup l_{-1} \cup l_{\frac{1}{2}} \cup l_2$ and the sets $\{(M_5(\alpha), E(M_5(\alpha))\}$ are now easily computed. The last step is to reduce the size of $\{(M_5(\alpha), E(M_5(\alpha))\}$ using Theorem 1.0.7. The result is shown in Tables 1.3.1 and 1.3.2.

This completes the proof of the main content of Corollary 1.3.1; the final assertion of Corollary 1.3.1 is easily proved by noting that each α in Tables 1.3.1 and 1.3.2 contains a slope found in $[\![(\frac{1}{2})_0)]\!]$ and so, by Theorem 2.1.1, each α with $e\big(M_5(\alpha)\big) > 3$ factors through the exterior of the chain link L from Figure 1.3.1. $\square$

The elementary techniques used to prove Corollary 1.3.1 can obviously be applied to describe all $E(M_5(\alpha))$ when α factors through M_4 but α does not factor through M_3.

PROOF OF COROLLARY 1.3.2. The argument proceeds exactly as in the proof of Corollary 1.3.1. We let $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \varnothing)$ be a non-exceptional filling instruction on M_5 factoring through M_4 but not factoring through M_3. Thus some α_i belongs to $\{-1, \frac{1}{2}, 2\}$ and we use Theorem 1.0.7 to assume $\alpha_0 = -1$ without loss of generality. The conditions of α being non-exceptional and not factoring through M_3 mean $\alpha_1 \notin \{0, 1, \infty, -1, -2, -\frac{1}{2}\}$ and $\alpha_2, \alpha_3 \notin \{0, 1, \infty, -1, \frac{3}{2}, 3\}$.

If β is an exceptional slope on $M_5(\alpha)$, then, by Theorem 1.1.3, $\beta \in \{-1, 0, 1, \infty\}$, or (α, β) factors through M_3, or $M_5(\alpha, \beta) = m_2$. This implies that β is in one of $[\![(1)]\!]$, $[\![(-1)]\!]$, $[\![(-2)]\!]$. Theorem 1.1.3 implies that every $\beta \in \{-1, 0, 1, \infty\}$ is an exceptional slope on $M_5(\alpha)$.

Every slope in $\{[\![(-1)]\!], [\![(-2)]\!]\}$ is examined individually as in the proof of Corollary 1.3.1 to obtain a complete list of all $\big(M_5(\alpha), E(M_5(\alpha))\big)$ pairs that have

$\alpha = \big(-1, \frac{p}{q}, \frac{r}{s}, \frac{u}{v}, \varnothing\big)$, $M_5(\alpha)$ hyperbolic not factoring through M_3 with $e\big(M_5(\alpha)\big) > 4$. The (α, β) coming from m_2 are given by $(-1, -3, -2, -2, -3)$, $(-1, \frac{1}{3}, \frac{4}{3}, \frac{4}{3}, \frac{1}{3})$, $(-1, -\frac{1}{3}, 4, \frac{2}{3}, 3)$, $(-1, 3, \frac{2}{3}, 4, -\frac{1}{3})$.

Theorem 1.0.7 is used to reduce the number of cases and the results are shown in Table 1.3.3. $\Box$

Remark It turns out that no α in Tables 1.3.1-1.3.3 has $\{-\frac{1}{2}, -2\}$, or $\{\frac{1}{2}, 2\}$ as a subset of $E(M_5(\alpha))$. This helps to control the size of $E(M_5(\alpha))$ (see Theorem 0.0.4).

CHAPTER 3

The magic manifold

The definition of the magic manifold was recalled in the introduction of this thesis: it is the exterior of the 3-chain link 3CL (introduced in Figure 0.0.2) and it is denoted by M_3. The Magic manifold plays an enormous part in the classification of the exceptional Dehn surgeries of the minimally twisted five chain link (see Theorems 1.1.1 and 1.1.3).

In this chapter we look at Problem 2 from the Gordon program mentioned in the introduction of this thesis for $M_5(\alpha)$ when α factors through M_3. The main result of this chapter is Theorem 3.2.6, that gives a complete description of all pairs of the form $(M_3(f); \alpha_1, \alpha_2)$ of type $(\mathcal{C}_1, \mathcal{C}_2)$ with $\mathcal{C}_i \in \{S^H, S, T^H, T\}$ and $\Delta(\alpha, \beta) = \Delta_0(\mathcal{C}_1, \mathcal{C}_2)$ when f is a filling instruction on M_3 with exactly two slopes (definitions are contained in the introduction of this thesis).

Part of this problem has already been completed; the sets of exceptional slopes and the corresponding fillings for all manifolds of the form $M_3(\frac{p}{q}, \frac{r}{s})$ are described in the appendix of [**MP**] when $e(M_3(\frac{p}{q}, \frac{r}{s})) > 5$. In this Chapter we consider all exceptional sets. All the results of this chapter will be elementary consequences of Theorem 3.0.1.

The structure of the chapter is as follows; in Section 3.1 we give a new description of the manifolds obtained as exceptional fillings of M_3. Section 3.2 contains the proof of Theorem 3.2.6, the main result of this chapter and deals with Problem 2 from the Gordon program when $\mathcal{C}_i$ is allowed to be of type Z, that is, the class of small Seifert spaces fibered over the sphere with exactly 3 exceptional fibres. In the final section we introduce a measure on how the fibres of the Seifert pieces in graph manifolds intersect, which we call the “intersection index”, and we list the sets of exceptional slopes and fillings of all hyperbolic manifolds of the form $M_3(\frac{p}{q}, \frac{r}{s})$ that admit a graph manifold filling with intersection index greater than 1.

We remind the reader that the link symmetries of 3CL induces an S_3-action on filling instructions for M_3, and that we will consider filling instructions on M_3 up to the action of all possible permutations of slopes (see Section 1.1). We now state the classification of the exceptional Dehn fillings of the Magic manifold carried out by Martelli and Petronio [**MP**].

THEOREM 3.0.1. (Martelli-Petronio) *A filling instruction on M_3 is exceptional if and only if it contains an isolated filling instruction. A complete list of all isolated exceptional filling instructions of M_3 is described in Tables 3.0.1-3.0.2 along with the JSJ decomposition of the corresponding fillings. Moreover, a complete list of all exceptional filling instructions of M_3 is described in Tables 3.0.3-3.0.5 along with the JSJ decomposition of the corresponding fillings.*

p/q	r/s	$M_3\left(\frac{p}{q}, \frac{r}{s}\right)$
∞	-	$T \times [0,1]$
0	-	$\big(D, (2,1), (3,1)\big) \bigcup_{\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}} \big(P \times S^1\big)$
1	-	$\big(A, (2,1)\big)$
2	-	$\big(A, (3,1)\big)$
3	-	$\big(A, (2,1)\big) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \big(A, (2,1)\big)$
-1	-1	$\big(P \times S^1\big) / _{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$
$\frac{3}{2}$	$\frac{5}{2}$	$\big(D, (2,1), (3,1)\big) \bigcup_{\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}} \big(A, (2,1)\big)$
4	$\frac{1}{2}$	$\big(D, (2,1), (3,1)\big) \bigcup_{\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}} \big(A, (2,1)\big)$

TABLE 3.0.1. All non-closed isolated exceptional filling instructions for M_3 and the corresponding fillings of M_3.

$\frac{p}{q}$	$\frac{r}{s}$	$\frac{u}{v}$	$M(\frac{p}{q}, \frac{r}{s}, \frac{u}{v})$
5	5	$\frac{1}{2}$	$\big(A,(2,1)\big)/_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}$
4	4	$\frac{2}{3}$	$\big(A,(2,1)\big)/_{\begin{pmatrix}1&1\\1&0\end{pmatrix}}$
4	$\frac{3}{2}$	$\frac{3}{2}$	$T_{\begin{pmatrix}-3&1\\-1&0\end{pmatrix}}$
4	$\frac{1}{3}$	-1	$\big(D,(2,1),(2,1)\big)\bigcup_{\begin{pmatrix}0&1\\-1&-1\end{pmatrix}}\big(D,(2,1),(3,1)\big)$
$\frac{8}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	$\big(D,(2,1),(2,1)\big)\bigcup_{\begin{pmatrix}0&1\\-1&-1\end{pmatrix}}\big(D,(2,1),(3,1)\big)$
$\frac{5}{2}$	$\frac{5}{2}$	$\frac{4}{3}$	$\big(A,(2,1)\big)/_{\begin{pmatrix}2&1\\1&0\end{pmatrix}}$
$\frac{5}{2}$	$\frac{5}{3}$	$\frac{5}{3}$	$\big(D,(2,1),(2,1)\big)\bigcup_{\begin{pmatrix}-1&1\\0&-1\end{pmatrix}}\big(D,(2,1),(3,1)\big)$
$\frac{7}{3}$	$\frac{7}{3}$	$\frac{3}{2}$	$\big(A,(2,1)\big)/_{\begin{pmatrix}1&1\\1&0\end{pmatrix}}$
-1	-2	-2	$\big(S^2,(2,1),(3,1),(7,1),-1\big)$
-1	-2	-3	$\big(S^2,(2,1),(4,1),(5,1),-1\big)$
-1	-2	-4	$\big(S^2,(3,1),(3,1),(4,1),-1\big)$
-1	-2	-5	$\big(D,(2,1),(2,1)\big)\bigcup_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}\big(D,(2,1),(3,1)\big)$
-1	-3	-3	$\big(D,(2,1),(2,1)\big)\bigcup_{\begin{pmatrix}1&2\\0&-1\end{pmatrix}}\big(D,(2,1),(3,1)\big)$
-2	-2	-2	$\big(D,(2,1),(2,1)\big)\bigcup_{\begin{pmatrix}2&3\\-1&-2\end{pmatrix}}\big(D,(2,1),(3,1)\big)$

TABLE 3.0.2. All closed isolated exceptional filling instructions and the corresponding fillings of M_3.

$\frac{p}{q}$	$\frac{r}{s}$	M
∞	$\frac{r}{s}$	$D \times S^1$
3	1	$\mathbb{RP}^3 \# (D \times S^1)$
	2	$D \times S^1$
	3	$\big(S, (2,1)\big)$
	$1 - \frac{1}{n}, \neq 2$	$\big(D, (2,1), (1+2n, 2)\big)$
	$\neq 1, \neq 1 - \frac{1}{n}$	$\big(D, (2,1), (s+r, s)\big) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \big(A, (2,1)\big)$
2	2	$L(3,1) \# (D \times S^1)$
	$2 - \frac{1}{n}$	$D \times S^1$
	$\neq 2, \neq 2 - \frac{1}{n}$	$\big(D, (3,1), (2s+r, s)\big)$
1	3	$\mathbb{RP}^3 \# (D \times S^1)$
	$3 - \frac{1}{n}$	$D \times S^1$
	$\neq 3, \neq 3 - \frac{1}{n}$	$\big(D, (2,1), (3s+r, s)\big)$
0	$\in \mathbb{Z}$	$\big(D, (2,1), (3,1)\big)$
	$\notin \mathbb{Z}$	$\big(D, (2,1), (3,1)\big) \bigcup_{\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}} \big(A, (s, -r)\big)$

TABLE 3.0.3. All non-isolated exceptional filling instructions and the corresponding fillings of M_3 with one boundary component.

p/q	r/s	u/v	$M_3\left(\frac{p}{q}, \frac{r}{s}, \frac{t}{u}\right)$
∞	$\frac{r}{s}$	$\frac{u}{v}$ with $u\nu - v\mu = 1$	$L(ur-vs,\, r\nu-s\mu)$
3	-1 or $\frac{5}{3}$	r/s	$(K,1)$
	1	$\frac{u}{v}$	$\mathbb{RP}^3 \# L(3v-u,\, v)$
	2	$\frac{u}{v}$	$L(5u-7v,\, 2u-3v)$
	$1-\frac{1}{n}$ $\neq 2$	$1-\frac{1}{m}$	$L\big((2n+1)(2m+1)-4,\, (2n+1)m-2\big)$
		$\neq 1,\ \neq 1-\frac{1}{m}$	$\big(S^2, (2,1), (2n+1,-2), (v-u,v)\big)$
	3	$\neq 1,\ \neq 1-\frac{1}{m}$	$\big(\mathbb{RP}^2, (2,1), (v-u,v)\big)$
	$\neq 1,\ \neq 3$ $\neq 1-\frac{1}{n}$	$\neq 1,\ \neq 3$ $\neq 1-\frac{1}{m}$ $\neq \frac{r}{s}$ if $\frac{r}{s} \in \{-1, \frac{5}{3}\}$	$\big(D, (2,1), (s-r,s)\big) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \big(D, (2,1), (v-u,v)\big)$
2	2	$\frac{u}{v}$	$L(3,1) \# L(2v-u,\, v)$
	$2-\frac{1}{n}$	$\frac{u}{v}$	$L\big(3n(u-2v)-2u+v,\, n(u-2v)-u+v\big)$
	$\neq 2$ $\neq 2-\frac{1}{n}$	$\neq 2$ $\neq 2-\frac{1}{m}$	$\big(S^2, (3,2), (r-2s,s), (u-2v,v)\big)$
1	3	$\frac{u}{v}$	$\mathbb{RP}^3 \# L(3v-u,\, v)$
	$3-\frac{1}{n}$	$\frac{u}{v}$	$L\big(2n(u-3v)-u+v,\, n(u-3v)-v\big)$
	$\neq 3$ $\neq 3-\frac{1}{n}$	$\neq 3$ $\neq 3-\frac{1}{m}$	$\big(S^2, (2,1), (r-3s,s), (u-3v,v)\big)$
0	n	$4-n$	$\mathbb{RP}^3 \# L(3,1)$
		$4-n-\frac{1}{m}$	$L(6m-1,\, 2m-1)$
		$\neq 4-n$ $\neq 4-n-\frac{1}{m}$	$\big(S^2, (2,-1), (3,1), ((4-n)v-u,\, v)\big)$
	$n-\frac{1}{2}$	$n+\frac{9}{2}$	$\big(\mathbb{RP}^2, (2,1), (3,1), -1\big)$
	$\notin \mathbb{Z}$	$\notin \mathbb{Z}$ $\neq \frac{r}{s}+4$ if $\lvert s\rvert = 2$	$\big(D, (s,2s-r), (v,2v-u)\big) \bigcup_{\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}} \big(D, (2,1), (3,1)\big)$

TABLE 3.0.4. All closed non-isolated exceptional filling instructions and the corresponding fillings of M_3, part 1 of 2.

p/q	r/s	u/v	M
-1	-1	n	$T_{\begin{pmatrix}1-n & 1\\ -1 & 0\end{pmatrix}}$
		$\notin \mathbb{Z}$	$\big(A, (v, v-u)\big) / \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$
$3/2$	$5/2$	2	$\mathbb{RP}^3$
		1	$L(13, 5)$
		0	$\big(\mathbb{RP}^2, (2,1), (3,1), -1\big)$
		$2-\frac{1}{n}$ $\neq 1$	$\big(S^2, (2,1), (3,-1), (2n-1, 2)\big)$
		$\neq 0$ $\neq 2$ $\neq 2-\frac{1}{n}$	$\big(D, (2,1), (3,1)\big) \bigcup_{\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}} \big(D, (2,1), (2v-u, v)\big)$
4	$1/2$	1	$L(11, 3)$
		$\frac{1}{2}$	$\big(\mathbb{RP}^2, (2,1), (3,1), -1\big)$
		0	$L(13, 5)$
		n $\neq 1$ $\neq 0$	$\big(S^2, (2,1), (3,-1), (1-2n, 2)\big)$
		$\notin \mathbb{Z} \cup \{\frac{1}{2}\}$	$\big(D, (2,1), (3,1)\big) \bigcup_{\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}} \big(D, (2,1), (v, -u)\big)$

TABLE 3.0.5. All closed non-isolated exceptional filling instructions and the corresponding fillings of M_3, part 2 of 2.

Remark Tables 3.0.1 and 3.0.2 come from [**MP**] and agree with the original tables up to a couple of obvious typos and the reversal of the orientation.

3.1. From filling instructions to fillings

There are 21 isolated exceptional fillings of M_3, as shown in Tables 3.0.1 and 3.0.2. We will show in this section that certain isolated exceptional fillings are easily recognised as fillings of F or of $M_3(-1,-1)$. Fillings of F can be obtain using Proposition 1.2.3 and fillings of $M_3(-1,-1)$ can be obtained using Theorem 3.0.1. This leads to a complete description of the JSJ decomposition of 14 of the 21 isolated exceptional manifolds.

Proposition 3.1.1. *For each exceptional filling instruction f on M_3 not in*

$$\{(-1,-2,-2),(-1,-2,-3),(-1,-2,-4),(-1,-2,-5),(-1,-3,-3),(-2,-2,-2)\},$$

$M_3(f)$ can be expressed as some filling of F or of $M_3(-1,-1)$ and the correspondence is found in Table 3.1.1.

$M_3\big(\frac{p}{q},\frac{r}{s},\infty\big)=F\big(-\frac{1}{2},\frac{q}{q-p},\frac{1}{3},\frac{r}{s}\big)$	$M_3\big(\frac{p}{q},\frac{r}{s},0\big)=F\big(\frac{s}{s-r},2,\frac{q}{3q-p},-3\big)$
$M_3\left(\frac{p}{q},\frac{r}{s},1\right)=F\big(\frac{p-3q}{q},-1,-2,\frac{r-2s}{s}\big)$	$M_3\big(\frac{p}{q},\frac{r}{s},2\big)=F\big(\frac{s}{r},\frac{2q-p}{p-q},-\frac{1}{2},3\big)$
$M_3\big(\frac{p}{q},\frac{r}{s},3\big)=F\big(-2,-2,1,\frac{p-q}{q},\frac{r-s}{s}\big)$	$M_3\big(\frac{p}{q},\frac{3}{2},\frac{5}{2}\big)=F\big(-3,\frac{p-2q}{p-q},2,-2\big)$
$M_3\big(4,\frac{1}{2},\frac{p}{q}\big)=F\big(2,\frac{3}{2},\frac{q}{p-q},-2\big)$	$M_3\big(-1,\frac{1}{3},4\big)=M_3\big(\frac{3}{2},\frac{3}{2},\frac{8}{3}\big)$ $=F\big(-3,\frac{2}{3},2,-2\big)$
$M_3\big(\frac{5}{2},\frac{5}{3},\frac{5}{3}\big)=F\big(2,\frac{2}{3},\frac{2}{3},-2\big)$	
$M_3\big(\frac{3}{2},\frac{7}{3},\frac{7}{3}\big)=M_3\big(4,4,\frac{2}{3}\big)$ $=M_3\big(-1,-1,\frac{3}{2}\big)$	$M_3\big(5,5,\frac{1}{2}\big)=M_3\big(-1,-1,\frac{1}{2}\big)$
$M_3\big(\frac{5}{2},\frac{5}{2},\frac{4}{3}\big)=M_3\big(-1,-1,\frac{5}{2}\big)$	$M_3\big(4,\frac{3}{2},\frac{3}{2}\big)=M_3\big(-1,-1,4\big)$

Table 3.1.1. Homeomorphisms between fillings of M_3 and of F or $M_3(-1,-1)$.

Proof. We prove the equalities in Table 3.1.1 using Theorems 1.0.7 and 2.1.1, and include a subscript to indicate which of the these results is being employed.

We start with the case where f is a filling instruction of M_3 and $M_3(f)$ is homeomorphic to a filling of F:

$$\begin{aligned} M_3\big(\tfrac{p}{q},\tfrac{r}{s},\infty\big) &\underset{2.1.1}{=} M_5\big(-1,-2,\tfrac{p-q}{q},\tfrac{r}{s},\infty\big)\\ &\underset{1.0.7}{=} M_5\big(\tfrac{1}{2},1,\tfrac{p-2q}{p-q},\tfrac{1}{3},\tfrac{r}{s}\big) \underset{2.1.1}{=} F\big(-\tfrac{1}{2},\tfrac{q}{q-p},\tfrac{1}{3},\tfrac{r}{s}\big)\end{aligned} \tag{3.1}$$

$$\begin{aligned} M_3\big(\tfrac{p}{q},\tfrac{r}{s},0\big) &\underset{2.1.1}{=} M_5\big(\tfrac{p-2q}{q},-1,-2,\tfrac{r-s}{s},0\big)\\ &\underset{1.0.7}{=} M_5\big(\tfrac{r-2s}{r-s},2,\tfrac{q}{3q-p},-2,1\big) \underset{2.1.1}{=} F\big(\tfrac{s}{s-r},2,\tfrac{q}{3q-p},-3\big)\end{aligned} \tag{3.2}$$

$$M_3\big(\tfrac{p}{q},\tfrac{r}{s},1\big) \underset{2.1.1}{=} M_5\big(\tfrac{p-2q}{q},-1,-2,\tfrac{r-s}{s},1\big) \underset{2.1.1}{=} F\big(\tfrac{p-3q}{q},-1,-2,\tfrac{r-2s}{s}\big) \tag{3.3}$$

$$\begin{aligned} &M_3\big(\tfrac{p}{q},\tfrac{r}{s},2\big) \underset{2.1.1}{=} M_5\big(0,-1,-2,\tfrac{p-q}{q},\tfrac{r}{s}\big) \\ &\underset{1.0.7}{=} M_5\big(\tfrac{s}{r},\tfrac{q}{p-q},1,\tfrac{1}{2},3\big) \underset{2.1.1}{=} F\big(\tfrac{s}{r},\tfrac{2q-p}{p-q},-\tfrac{1}{2},3\big) \end{aligned} \tag{3.4}$$

$$M_3\big(\tfrac{p}{q},\tfrac{r}{s},3\big) \underset{2.1.1}{=} M_5\big(-1,\tfrac{p-q}{q},\tfrac{r-s}{s},-1,1\big) \underset{2.1.1}{=} F\big(-2,-2,1,\tfrac{p-q}{q},\tfrac{r-s}{s}\big) \tag{3.5}$$

$$\begin{aligned} &M_3\big(\tfrac{p}{q},\tfrac{3}{2},\tfrac{5}{2}\big) \underset{2.1.1}{=} M_5\big(\tfrac{1}{2},-2,-1,\tfrac{1}{2},\tfrac{p}{q}\big) \underset{1.0.7}{=} M_5\big(-1,\tfrac{1}{2},\tfrac{p}{q},\tfrac{1}{3},-1\big) \\ &\underset{2.1.1}{=} M_5\big(0,\tfrac{1}{2},\tfrac{p-q}{q},-1,\tfrac{1}{3}\big) \underset{1.0.7}{=} M_5\big(1,-2,\tfrac{p-2q}{p-q},2,-1\big) \underset{2.1.1}{=} F\big(-3,\tfrac{p-2q}{p-q},2,-2\big) \end{aligned} \tag{3.6}$$

$$\begin{aligned} &M_3\big(4,\tfrac{1}{2},\tfrac{p}{q}\big) \underset{2.1.1}{=} M_5\big(2,\tfrac{1}{2},\tfrac{p-q}{q},-2,-1\big) \underset{1.0.7}{=} M_5\big(\tfrac{1}{2},\tfrac{p-q}{q},-1,-1,\tfrac{3}{2}\big) \\ &\underset{2.1.1}{=} M_5\big(-\tfrac{1}{2},-1,\tfrac{p-q}{q},0,\tfrac{3}{2}\big) \underset{1.0.7}{=} M_5\big(3,\tfrac{3}{2},\tfrac{q}{p-q},-1,1\big) \underset{2.1.1}{=} F\big(2,\tfrac{3}{2},\tfrac{q}{p-q},-2\big) \end{aligned} \tag{3.7}$$

$$\begin{aligned} &M_3\big(-1,\tfrac{1}{3},4\big) \underset{2.1.1}{=} M_5\big(-2,-1,-\tfrac{5}{3},-1,3\big) \underset{1.0.7}{=} M_5\big(\tfrac{1}{2},\tfrac{8}{3},-\tfrac{1}{2},-1,-2\big) \\ &\underset{2.1.1}{=} M_3\big(\tfrac{3}{2},\tfrac{3}{2},\tfrac{8}{3}\big) \underset{2.1.1}{=} M_5\big(-1,\tfrac{2}{3},-1,\tfrac{1}{2},\tfrac{1}{2}\big) \underset{1.0.7}{=} M_5\big(2,-1,\tfrac{1}{2},\tfrac{3}{2},-1\big) \\ &\underset{2.1.1}{=} M_3\big(4,\tfrac{3}{2},\tfrac{5}{2}\big) \underset{(3.6)}{=} F\big(-3,\tfrac{2}{3},2,-2\big) \end{aligned} \tag{3.8}$$

$$\begin{aligned} &M_3\big(\tfrac{5}{2},\tfrac{5}{3},\tfrac{5}{3}\big) \underset{2.1.1}{=} M_5\big(\tfrac{3}{2},-1,-\tfrac{1}{3},-1,\tfrac{2}{3}\big) \underset{1.0.7}{=} M_5\big(-2,-\tfrac{1}{2},4,\tfrac{1}{2},-1\big) \\ &\underset{2.1.1}{=} M_3\big(\tfrac{1}{2},4,\tfrac{5}{2}\big) \underset{(3.7)}{=} F\big(2,\tfrac{3}{2},\tfrac{2}{3},-2\big) \end{aligned} \tag{3.9}$$

We now turn to the case where f is a filling instruction of M_3 and $M_3(f)$ is homeomorphic to a filling of $M_3(-1,-1)$:

$$\begin{aligned} &M_3\big(\tfrac{3}{2},\tfrac{7}{3},\tfrac{7}{3}\big) \underset{2.1.1}{=} M_5\big(-2,-1,\tfrac{1}{3},\tfrac{7}{3},\tfrac{1}{2}\big) \underset{1.0.7}{=} M_5\big(-1,-\tfrac{4}{3},-1,3,3\big) \\ &\underset{2.1.1}{=} M_3\big(4,4,\tfrac{2}{3}\big) \underset{2.1.1}{=} M_5\big(2,\tfrac{2}{3},3,-2,-1\big) \underset{1.0.7}{=} M_5\big(-\tfrac{1}{2},-1,-2,-2,-1\big) \\ &\underset{2.1.1}{=} M_3\big(-1,-1,\tfrac{3}{2}\big) \end{aligned} \tag{3.10}$$

$$\begin{aligned} &M_3\big(5,5,\tfrac{1}{2}\big) \underset{2.1.1}{=} M_5\big(3,\tfrac{1}{2},4,-2,-1\big) \underset{1.0.7}{=} M_5\big(-\tfrac{1}{2},-1,-3,-1,-2\big) \\ &\underset{2.1.1}{=} M_3\big(-1,-1,\tfrac{1}{2}\big) \end{aligned} \tag{3.11}$$

$$M_3\big(\tfrac{5}{2},\tfrac{5}{2},\tfrac{4}{3}\big) \underset{2.1.1}{=} M_5\big(\tfrac{1}{2},-1,\tfrac{3}{2},\tfrac{1}{3},-1\big) \underset{1.0.7}{=} M_5\big(-2,-2,-1,\tfrac{1}{2},-1\big) \underset{2.1.1}{=} M_3\big(-1,-1,\tfrac{5}{2}\big) \tag{3.12}$$

$$M_3\big(4,\tfrac{3}{2},\tfrac{3}{2}\big) \underset{2.1.1}{=} M_5\big(4,\tfrac{1}{2},-2,-1,-\tfrac{1}{2}\big) \underset{1.0.7}{=} M_5\big(-1,-2,-3,-1,-3\big) \underset{2.1.1}{=} M_3\big(4,-1,-1\big) \tag{3.13}$$

This completes the proof. □

Remark The identities between fillings of M_3 in 3.10-3.13 are obtained by going through M_5, but infact it is also possible to see these identifications directly, see [**MP**] for details.

Example 3.1.2. Consider the case $M_3(0,p/q,r/s)$. From Table 3.1.1 and Proposition 1.2.3 we get

$$M_3(0,p/q,r/s) = \big(D,(s,s-r),(q,3q-p)\big) \bigcup_{\begin{pmatrix}0&1\\1&0\end{pmatrix}} \big(D,(2,1),(3,-1)\big), \tag{3.14}$$

while from Table 3.0.4 we get

$$M_3(0,p/q,r/s) = \big(D,(s,2s-r),(q,2q-p)\big) \bigcup_{\begin{pmatrix}0&1\\-1&-1\end{pmatrix}} \big(D,(2,1),(3,1)\big). \tag{3.15}$$

We now use Lemma 2.2.1 to show that (3.14) and (3.15) are consistent with each other. By (2.8) and (2.11), the graph manifold presentation of $M_3(0,p/q,r/s)$ in (3.14) is equal to

$$\big(D,(s,2s-r),(q,2q-p)\big) \bigcup_{\begin{pmatrix}0&1\\1&1\end{pmatrix}} \big(D,(2,-1),(3,-1)\big).$$

A reversal of orientation on the second Seifert piece reverses the orientation on the boundary which is accounted for in the gluing matrix by post multiplying by $\begin{pmatrix}1&0\\0&-1\end{pmatrix}$. The result is that the graph manifold in (3.14) is equal to that in (3.15).

Similar straightforward applications of Proposition 1.2.3 and Lemma 2.2.1 show:

Proposition 3.1.3. *Table 3.1.1 and Proposition 1.2.3 are consistent with Tables 3.0.1-3.0.5.*

3.2. Knot exteriors and cyclic fillings of M_3

All filling instructions f for M_3 with $M_3(f) = L(p, q)$ can be found within Tables 3.0.1-3.0.5 for $p \geq 1$. By Theorem 3.0.1, $M_3(f)$ is non-hyperbolic if and only if f contains an isolated exceptional filling instruction. Thus, using these tables, we can describe all (f, α) with $M_3(f)$ a hyperbolic manifold with one boundary component and $M_3(f)(\alpha)$ an exceptional lens space filling. The resulting (f, α) are shown in Table 3.2.1

$f = (\frac{p}{q}, \frac{r}{s})$	α	$M_3(f)(\alpha)$	Restrictions
$(\frac{p}{q}, \frac{r}{s})$	∞	$L(pr-qs,\, r\mu-s\nu)$ where $\lvert p\nu - q\mu\rvert = 1$	$\frac{p}{q}, \frac{r}{s} \neq 0, 1, 2, 3, \infty$, $\{\frac{p}{q}, \frac{r}{s}\} \neq \{\frac{3}{2}, \frac{5}{2}\}, \{\frac{1}{2}, 4\}$, $\frac{p}{q} = \frac{r}{s} \Rightarrow \frac{p}{q} \neq -1$
$(1 - \frac{1}{n}, 1 - \frac{1}{m})$	3	$L((2n+1)(2m+1)-4,\, (2n+1)m-2)$	$n, m \neq 0, \pm 1$
$(2 - \frac{1}{n}, \frac{r}{s})$	2	$L(3n(r-2s)-2r+s,\, n(r-2s)-r+s)$	$n \neq 0, \pm 1, n = 2 \Rightarrow \frac{r}{s} \neq \frac{5}{2}$, $n = -2 \Rightarrow \frac{r}{s} \neq \frac{3}{2}$
$(3 - \frac{1}{n}, \frac{r}{s})$	1	$L(2n(r-3s)-r+s,\, n(r-3s)-s)$	$n \neq 0, 1, n = 2 \Rightarrow \frac{r}{s} \neq \frac{3}{2}$, $n = -1 \Rightarrow \frac{r}{s} \neq \frac{1}{2}$
$(n, 4 - n - \frac{1}{m})$	0	$L(6m-1,\, 2m-1)$	$n \neq 0, 1, 2, 3, m \neq 0$, $n = 4 \Rightarrow m \neq -2$

TABLE 3.2.1. All (f, α) with $M_3(f)$ hyperbolic and $M_3(f)(\alpha)$ a lens space.

Note that $M_3(f)$ is a hyperbolic knot complement in S^3 if and only if f consists of two slopes, f does not contain an isolated exceptional filling instruction, and there exists an α such that $M_3(f)(\alpha) = S^3$. This allows us to find all hyperbolic knot complements in S^3 of the form $M_3(f)$, and to look at which knots complements in S^3 of the form $M_3(f)$ have fillings of the form $L(p, q)$ with $p > 1$:

PROPOSITION 3.2.1. *Table 3.2.2 describes all (f, α) with $M_3(f)$ a hyperbolic manifold with one boundary component and α an exceptional slope on $M_3(f)$ with $M_3(f)(\alpha) = S^3$. Moreover, every $M_3(\frac{p}{q}, \frac{r}{s})$ homeomorphic to a knot complement in S^3 with a lens space filling is described in Table 3.2.3.*

α	$f = (\frac{p}{q}, \frac{r}{s})$	restrictions
∞	$rp - qs = \pm 1$	$\frac{p}{q}, \frac{r}{s} \notin \{0,1,2,3\}$ $\{\frac{p}{q}, \frac{r}{s}\} \neq \{\frac{3}{2}, \frac{5}{2}\}, \{\frac{1}{2}, 4\}$, $\frac{p}{q} = \frac{r}{s} \Rightarrow \frac{p}{q} \neq -1$
2	$f = \left(\frac{2n-1}{n}, \frac{(6n-1)k+\epsilon(2n-1)}{(3n-2)k+\epsilon(n-1)}\right)$, $n,\ k \in \mathbb{Z}$, $\epsilon = \pm 1$	$n \neq 0, \pm 1$
1	$f = \left(\frac{3n-1}{n}, \frac{(6n-1)k+\epsilon(3n-2)}{(2n-1)k+\epsilon(n-1)}\right)$, $n,\ k \in \mathbb{Z}$, $\epsilon = \pm 1$	$n \neq 0, 1$

TABLE 3.2.2. All (f, α) with $M_3(f)$ a hyperbolic and $M_3(f)(\alpha) = S^3$.

(S^H, T^H)−pair	$f = (\frac{p}{q}, \frac{r}{s})$	restrictions
$(\infty, 3)$	$(\frac{n-1}{n}, \frac{n+1}{n})$, $n \in \mathbb{Z}$	$n \neq 0, \pm 1$
	$\left(\frac{n-1}{n}, \frac{1-n}{2-n}\right)$, $n \in \mathbb{Z}$	$n \neq 0, 1, 2$
$(\infty, 2)$	$\left(2 - \frac{1}{n}, \frac{nk-\epsilon}{(2n-1)k-2\epsilon}\right)$, $n, k \in \mathbb{Z}$, $\epsilon = \pm 1$	$n \neq 0, \pm 1$
$(\infty, 1)$	$\left(3 - \frac{1}{n}, \frac{nk-\epsilon}{(3n-1)k-3\epsilon}\right)$, $n, k \in \mathbb{Z}$, $\epsilon = \pm 1$	$n \neq 0, 1$
$(2, \infty)$	$\left(\frac{2n-1}{n}, \frac{(6n-1)k+\epsilon(2n-1)}{(3n-2)k+\epsilon(n-1)}\right)$, $n, k \in \mathbb{Z}$, $\epsilon = \pm 1$	$n \neq 0, \pm 1$
$(1, \infty)$	$\left(\frac{3n-1}{n}, \frac{(6n-1)k+\epsilon(3n-2)}{(2n-1)k+\epsilon(n-1)}\right)$, $n, k \in \mathbb{Z}$, $\epsilon = \pm 1$	$n \neq 0, 1$
$(2, 1)$	$\left(\frac{5}{2}, \frac{13k+5\epsilon}{8k+3\epsilon}\right)$, $n, k \in \mathbb{Z}$, $\epsilon = \pm 1$	-
$(1, 2)$	$\left(\frac{5}{2}, \frac{11k+4\epsilon}{3k+\epsilon}\right)$, $n, k \in \mathbb{Z}$, $\epsilon = \pm 1$	-

TABLE 3.2.3. All f with $M_3(f)$ hyperbolic admitting a lens space and S^3 filling.

PROOF. We will consider only pairs (f, α) contained in Table 3.2.1 as they include all the pairs with $M_3(f)(\alpha) = L(1, q) = S^3$ and $M_3(f)$ hyperbolic. We will now enumerate all such (f, α).

Suppose that $\alpha = 0$ is a filling instruction on a hyperbolic $M_3(f)$ giving S^3 as a result. Then by Table 3.2.1 we must have $f = (n, 4 - n - \frac{1}{m})$ and $M_3(n, 4 - n - \frac{1}{m})(0) = L(6m-1, 2m-1)$. This is S^3 only for $m = 0$, which is forbidden in Table 3.2.1 (because $M_3(f)$ is not hyperbolic in this case). So there is no (f, α) as described in the statement of Proposition 3.2.1 with $\alpha = 0$.

Similarly, if $\alpha = 3$ is a filling instruction on a hyperbolic $M_3(f)$ from Table 3.2.1 corresponding to an S^3 filling, then by Table 3.2.1 we must have $f = (1 - \frac{1}{n}, 1 - \frac{1}{m})$

and $M_3(1-\frac{1}{n}, 1-\frac{1}{m})(3) = L((2n+1)(2m+1)-4,\,(2n+1)m-2)$. In this case, $M_3(f)(3)$ is S^3 only for $(2n+1)(2m+1) = 3$ or 5, which implies that n or m is 0. These cases are forbidden in Table 3.2.1. So there is no (f, α) as described in the statement of Proposition 3.2.1 with $\alpha = 3$.

The case where $\alpha = \infty$ yields many solutions; we know that $M_3(\frac{p}{q}, \frac{r}{s})(\infty) = L(pr-qs,\, r\nu - s\mu)$ where $u\nu - v\mu = 1$. This is S^3 provided $pr - qs = \pm 1$ and $M_3(\frac{p}{q}, \frac{r}{s})$ is hyperbolic provided $(\frac{p}{q}, \frac{r}{s})$ does not contain an isolated filling instruction. We dismiss the cases when $(\frac{p}{q}, \frac{r}{s})$ contains an isolated exceptional filling instruction and the remaining cases are shown in Table 3.2.2.

For $\alpha = 2$ or $\alpha = 1$, one sees using Table 3.2.1, that characterizing the f with $M_3(f)$ hyperbolic and $M_3(f)(\alpha) = S^3$ is equivalent to finding all integer solutions of an equation of the form $a(n)x + b(n)y = c$, where $a(n)$ and $b(n)$ are elements of $\mathbb{Z}[n]$ of degree 1. The equations are given by;

$$(3n-2)u - (6n-1)v = \pm 1 \qquad (M_3(f)(2) \text{ case}) \tag{3.16}$$

$$(2n-1)u - (6n-1)v = \pm 1 \qquad (M_3(f)(1) \text{ case}) \tag{3.17}$$

For $a_i \in \mathbb{Z}$, the integer solutions of an equation $a_1x + a_2y = a_3$ are given by $\{(x_0 + \frac{b}{d}k, y_0 - \frac{a}{d}k)\}$ where $(x_0, y_0) \in \mathbb{Z}^2$ is any solution of $a_1x + a_2y = a_3$, $k \in \mathbb{Z}$, and d is the greatest common divisor of a_1 and a_2. For every n, $\gcd(3n-2, 6n-1) = \gcd(2n-1, 6n-1) = 1$ by the Euclidean algorithm, so the integer solutions of (3.16) and (3.17) are easy to write down; they can be found in Table 3.2.2. All $(\frac{p}{q}, \frac{r}{s})$ containing an exceptional isolated filling instruction are dismissed, which corresponding to imposing the restrictions on the $(\frac{p}{q}, \frac{r}{s})$'s found in Table 3.2.2. This completes the proof of the first assertion of Proposition 3.2.1.

The question of which hyperbolic knot complements in S^3 have cyclic surgeries is answered by finding all filling instructions $f = (\frac{p}{q}, \frac{r}{s})$ such that there exists distinct slopes α, β with (f, α) found in Table 3.2.2 and (f, β) found in Table 3.2.1. We can see from Tables 3.2.2 and 3.2.1 that all exceptional (S^H, T^H)-pairs of the form $(M_3(f); \alpha, \beta)$ must have $\alpha \in \{0, 1, 2, 3, \infty\}$ and $\beta \in \{1, 2, \infty\}$. We systematically examine all pairs $(M_3(f); \alpha, \beta)$ for such α and β.

Recall that no manifold has two S^3 surgeries, and that $\Delta_0(S^H, T^H) = 1$; thus when 2 or 1 is an S^3 filling of $M_3(f)$ then $M_3(f)(\infty)$ is a lens space filling. Moreover, we have $\Delta(1, 3) = \Delta(2, 0) = 2$ so the pairs of slopes $(1, 3)$ and $(2, 0)$ cannot correspond to an exceptional (S^H, T^H)-pair.

We now enumerate all $(M_3(\frac{p}{q},\frac{r}{s});1,0)$ of type (S^H,T^H). From Tables 3.2.1 and 3.2.2 we require integer solutions to

$$\left(\tfrac{p}{q},\tfrac{r}{s}\right)=\left(n',4-n'-\tfrac{1}{m}\right)=\left(\tfrac{3n-1}{n},\tfrac{(6n-1)k+\epsilon(3n-2)}{(2n-1)k+\epsilon(n-1)}\right)\text{ where }\epsilon=\pm 1$$

with $(\frac{p}{q},\frac{r}{s})$ containing no exceptional filling instruction. This means that $3-\frac{1}{n}=-n'$ or that $4+n'-\frac{1}{m}=3-\frac{1}{n}$. In the latter case n' is necessarily in $\{0,-1,-2\}$, which implies that f contains an exceptional filling instruction. In the former case $n=-1$ and $n'=4$, this implies that $-\frac{1}{m}=\frac{7k+5\epsilon}{3k+2\epsilon}$, which is easily seen to contain no integer solution. So, no $(M_3(f);1,0)$ is an exceptional $\left(S^H,T^H\right)$ pair. The (2, 3) case is similar.

For 1 or 2 to correspond to a lens space filling of $M_3(\frac{p}{q},\frac{r}{s})$ there is no restriction on $\frac{r}{s}$, and so there are many $(M_3(\frac{p}{q},\frac{r}{s});2,1)$'s and $(M_3(\frac{p}{q},\frac{r}{s});1,2)$'s of type (S^H,T^H). We see from Tables 3.2.1 and 3.2.2 that $(M_3(\frac{p}{q},\frac{r}{s});2,1)$ is of type (S^H,T^H) exactly when $2-\frac{1}{n}=3-\frac{1}{m}$, or $3-\frac{1}{m}=\frac{(6n-1)k+\epsilon(2n-1)}{(3n-2)k+\epsilon(n-1)}$. It is easy to verify that the latter equation contains no integer solutions, and we see that $(M_3(\frac{p}{q},\frac{r}{s});2,1)$ is of type $\left(S^H,T^H\right)$ if and only if $\left(\frac{p}{q},\frac{r}{s}\right)=\left(\frac{5}{2},\frac{13k+5\epsilon}{8k+3\epsilon}\right)$ for some $k\in\mathbb{Z},\epsilon=\pm 1$. Similarly we get that $(M_3(\frac{p}{q},\frac{r}{s});1,2)$ is of type (S^H,T^H) if and only if $\left(\frac{p}{q},\frac{r}{s}\right)=\left(\frac{5}{2},\frac{11k+4\epsilon}{4k+\epsilon}\right)$.

Lastly, we treat the case where ∞ corresponds to an S^3 filling; if $(M_3(f);\infty,0)$ is an exceptional (S^H,T^H)-pair then Table 3.2.2 implies that f must be of the form $(n,4-n-\frac{1}{m})$ with $n,m\in\mathbb{Z}$ (for $M_3(f)(0)$ to be of type T^H), and $4m=-1-\epsilon$ (for $M_3(f)(\infty)$ to be of type S^H); but $m\in\mathbb{Z}$, a contradiction. If $(M_3(f);\infty,3)$ is an exceptional (S^H,T^H)-pair on $M_3(f)$ then, from Tables 3.2.1 and 3.2.2 we require $f=(\frac{n-1}{n},\frac{m-1}{m})$ and $m+n=1-\epsilon$ (where $\epsilon=\pm 1$), writing m in terms of n and accounting for both $\epsilon=+1$ and $\epsilon=-1$, we find that this is possible only when $f=(\frac{n-1}{n},\frac{n+1}{n})$ or $(\frac{n-1}{n},\frac{1-n}{2-n})$. The cases $(M_3(f);\infty,2)$ and $(M_3(f);\infty,1)$ are solved by similar arguments and the solutions are found in Table 3.2.3. $\square$

Remark See Section 4.3 for a discussion of the connection of the above tables to the Berge knots.

Using Table 3.2.1 it is easy to find multiple lens space fillings on the same manifold.

Corollary 3.2.2. *Every hyperbolic $M_3(f)$ with one boundary component and three lens space fillings is given in Table 3.2.4, and every hyperbolic knot complement in S^3 of the form $M_3(f)$ with two lens space fillings is given in Table 3.2.5.*

$M_3(\frac{p}{q},\frac{r}{s})$	α	$M_3(\frac{p}{q},\frac{r}{s},\alpha)$	Restrictions
$M_3(4,-\frac{1}{n})$	∞	$L(n{+}4,\,-n{-}3)$	$n\neq 0$
	1	$L(7n{+}3,\,2n{+}1)$	
	0	$L(6n{-}1,\,2n{-}1)$	
$M_3(\frac{5}{2},\frac{r}{s})$	∞	$L(5r{-}2s,\,s{-}2r)$	$\frac{r}{s}\notin\{0,1,2,3,\infty,\frac{3}{2}\}$
	1	$L(11s{-}3r,\,2r{-}7s)$	
	2	$L(13s{-}8r,\,5s{-}3r)$	
$M_3(\frac{3}{2},1-\frac{1}{n})$	∞	$L(n{-}3,\,-1)$	$n\neq 0,\pm 1$
	2	$L(7n{+}4,\,8n{-}3)$	
	3	$L(6n{+}7,\,4n{+}4)$	

Table 3.2.4. Every hyperbolic knot exterior $M_3(f)$ with 3 lens space fillings.

Proof. The arguments are elementary and similar to those used in the proof of Proposition 3.2.1. We know $\Delta_0(T^H,T^H)=1$, and that ∞ is always a closed cyclic filling on $M_3(\frac{p}{q},\frac{r}{s})$. Thus each (T^H,T^H,T^H)-triple $(M_3(\frac{p}{q},\frac{r}{s}),\alpha,\beta,\gamma)$ must contain ∞ as a lens space filling. We enumerate all exceptional lens space pairs $(0,1)$, $(1,2)$, $(2,3)$ using the conditions imposed from Table 3.2.1. The result is that $(M_3(\frac{p}{q},\frac{r}{s});0,1)$ is an exceptional (T^H,T^H)-pair on $M_3(\frac{p}{q},\frac{r}{s})$ when $(\frac{p}{q},\frac{r}{s})=(4,-\frac{1}{m})$, $(M_3(\frac{p}{q},\frac{r}{s});1,2)$ is an exceptional (T^H,T^H)-pair when $(\frac{p}{q},\frac{r}{s})=(\frac{5}{2},\frac{r}{s})$, and $(M_3(\frac{p}{q},\frac{r}{s});2,3)$ is an exceptional (T^H,T^H)-pair when $(\frac{p}{q},\frac{r}{s})=(\frac{3}{2},1-\frac{1}{m})$.

The knot complements with three lens space fillings in Table 3.2.4 such that one exceptional filling is $L(\pm1,p)$ are the knot complements in S^3 with two lens space fillings. This is possible for $M_3(4,-\frac{1}{n})$ only when $n\in\{0,-3,-5\}$, possible for $M_3(\frac{3}{2},1-\frac{1}{n})$ only when $n\in\{-1,2,4\}$, and possible for $M_3(\frac{5}{2},\frac{r}{s})$ only when one of $|5r-2s|$, $|11s-3r|$, $|13s-8r|$ is 1. The solutions to these equations leaving $M_3(f)$ hyperbolic are the content of Table 3.2.5. □

$M_3(\frac{p}{q},\frac{r}{s})$	α	Space
$M_3(4,\frac{1}{5})$	∞	S^3
	2	$L(32,9)$
	3	$L(13,11)$
$M_3(4,\frac{1}{3})$	∞	S^3
	1	$L(18,5)$
	0	$L(19,7)$
$M_3(\frac{3}{2},\frac{1}{2})$	∞	S^3
	2	$L(18,13)$
	3	$L(19,12)$
$M_3(\frac{3}{2},\frac{3}{4})$	∞	S^3
	2	$L(32,129)$
	3	$L(31,20)$
$M_3(\frac{5}{2},\frac{2k+\epsilon}{5k+2\epsilon})$	∞	S^3
	1	$L(49k+19\epsilon,-31k-12\epsilon)$
	2	$L(49k+18\epsilon,19k+7\epsilon)$
$M_3(\frac{5}{2},\frac{11k-4\epsilon}{3k-\epsilon})$	∞	$L(49k-18\epsilon,-19k+5\epsilon)$
	1	S^3
	2	$L(-55k+19\epsilon,-18k+7\epsilon)$
$M_3(\frac{5}{2},\frac{13k-5\epsilon}{8k-3\epsilon})$	∞	$L(49k-19\epsilon,-18k+7\epsilon)$
	1	$L(49k-18\epsilon,-36k+11\epsilon)$
	2	S^3

Table 3.2.5. Every hyperbolic knot exterior in S^3 of the form $M_3(f)$ with 2 lens space fillings.

Remark We recall that $\Delta_0(T^H,T^H)=1$ and $\Delta_0(S^H,T^H)=1$. This implies no hyperbolic $M_3(f)$ has more than 3 exceptional lens space fillings, and no hyperbolic knot complement in S^3 has more than two lens space fillings, thus the examples in Tables 3.2.4 and 3.2.5 contain maximal numbers of lens space fillings.

We finish this section by looking at the knot complements in S^3 realising toroidal fillings at maximal distance from the S^3 filling. We recall that the knots complements in S^3 with an exceptional (S^H, T)-pair of slopes at maximal distance are exactly described by a collection of examples due to Eudave-Muñoz (see [**GL1**] and [**E**]) and are called the Eudave-Muñoz knots.

Proposition 3.2.3. *All Eudave-Muñoz knots of the form $M_3(f)$ are given in Table 3.2.6.*

$M_3(\frac{p}{q}, \frac{r}{s})$	(S^H, T) pair	restrictions
$M_3(4, \frac{\epsilon+k}{3\epsilon+4k})$	$(\infty, \frac{1}{2})$	$k \neq -\epsilon$
$M_3(\frac{5}{2}, \frac{\epsilon+2k}{2\epsilon+5k})$	$(\infty, \frac{3}{2})$	-
$M_3(\frac{3}{2}, \frac{\epsilon+2k}{\epsilon+3k})$	$(\infty, \frac{5}{2})$	$k \neq 0$
$M_3(3-\frac{1}{n}, \frac{(6n-1)k+\epsilon(3n-2)}{-(2n-1)k-\epsilon(n-1)})$	$(1,3)$	$n \neq \pm1, 0$
$M_3(2-\frac{1}{n}, \frac{(6n-1)k+\epsilon(2n-1)}{-(3n-2)k-\epsilon(n-1)})$	$(2,0)$	$n \neq \pm1, 0$

Table 3.2.6. Every $M_3(f)$ hyperbolic Eudave-Muñoz knot exterior.

Proof. We recall that $\Delta_0(S^H, T) = 2$ and that the Eudave-Muñoz knots are the class of knots in S^3 with a half-integral toroidal surgery. If $M_3(\frac{p}{q}, \frac{r}{s})$ is hyperbolic and α is a slope on $M_3(\frac{p}{q}, \frac{r}{s})$ with $M_3(\frac{p}{q}, \frac{r}{s})(\alpha) = S^3$ then $\alpha \in \{1, 2, \infty\}$ and the corresponding conditions on $\frac{p}{q}, \frac{r}{s}$ can be found in Table 3.2.2.

A slope α in an exceptional isolated filling instruction $(\frac{p}{q}, \frac{r}{s}, \alpha)$ with $\Delta(\infty, \alpha) = 2$ implies that $\alpha \in \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}\}$ (see Tables 3.0.1-3.0.5). We quickly see that such $(\frac{p}{q}, \frac{r}{s}, \alpha)$ do not satisfy the condition $pr - qs = \pm1$ necessary for ∞ to be an S^3 filling. Thus $(\frac{p}{q}, \frac{r}{s}, \alpha)$ necessarily strictly contains an isolated filling instruction, and we see that if $\alpha = \frac{1}{2}$ then one of $\frac{p}{q}$, $\frac{r}{s}$ is 4, and if $\alpha = \frac{3}{2}$ then one of $\frac{p}{q}$, $\frac{r}{s}$ is $\frac{5}{2}$, and if $\alpha = \frac{5}{2}$ then one of $\frac{p}{q}$, $\frac{r}{s}$ is $\frac{3}{2}$ (see Table 3.0.1-3.0.5). A choice on $\frac{p}{q}$ determines $\frac{r}{s}$; for example, if $\alpha = \frac{1}{2}$ and $\frac{p}{q} = 4$ then $4r - q = \epsilon$ in order for ∞ to be an S^3 filling. All $M_3(\frac{p}{q}, \frac{r}{s})$ with (∞, α) an exceptional (S^H, T) pair are easily computed and shown in Table 3.2.6. The cases with $\alpha = 1$ and $\alpha = 2$ corresponding to a S^3 filling of M_3 are similar. $\square$

Exceptional pairs at maximal distance. We recall for a hyperbolic manifold M with a specified toroidal boundary component τ, that $E_\tau(M)$ denotes the set of exceptional slopes on τ. Now we turn our attention to the general Gordon program where one of the fillings is not necessarily cyclic. We now look at the cases with $\mathcal{C}_i \neq Z, T^H$. The easiest case case involves reducible fillings:

Proposition 3.2.4. *No exceptional* (S,S) *or* (S^H,S) *pair of the form* $(M_3(f);\alpha,\beta)$ *exists. Further, all exceptional* (S,T)*-pairs of the form* $(M_3(f);\alpha,\beta)$ *with* $\Delta(\alpha,\beta) = \Delta_0(S,T)$ *and* $M_3(f)$ *having 1 boundary component are described by*

$$\{(M_3(n,4-n),0,3) : n \neq 0,1,2,3,4\}.$$

Moreover, for every $n \in \mathbb{Z}\backslash\{0,1,2,3,4\}$*, we have* $E(M_3(n,4-n)) = \{0,1,2,3,\infty\}$ *and* $\alpha = 0,1,2,3,\infty$ *is of type* S,Z,Z,T,T^H *respectively when* $n \neq -1$*, and of type* S,Z,T^H,T,T^H *respectively when* $n = -1$*.*

Proof. Table 3.0.4 completely describes hyperbolic manifolds of the form $M_3(f)$ with reducible filling; all hyperbolic manifolds $M_3(f)$ with a reducible filling α are completely described by $\{(M_3(n,4-n),\alpha=0),\ n \neq 0,1,2,3,4\}$. Using Tables 3.0.1-3.0.5 it is easy to see that $E(M_3(n,4-n)) = \{0,1,2,3,\infty\}$ for every n and that 3 is a toroidal filling, ∞ is a lens space filling, and that 1 and 2 are type Z fillings (unless $n=-1$ in which case 2 is of type T^H). It is now clear that no exceptional (S,S) or (S,S^H) pairs exist (cf Table 3.2.2). $\square$

We now consider all exceptional pairs $(M_3(f);\alpha,\beta)$ of types (T,T) and (T^H,T). For (T,T) it is known that $\Delta_0(T,T) = 8$ and that only the figure-8 and the figure-8 sister manifolds achieve this maximal distance [**GL1**]. It was noted in [**MP**] that the figure-8 knot complement is homeomorphic to $M_3(-1,-2)$, and that the sister figure-8 knot is homeomorphic to $M_3(-1,4) = M_3(4,\frac{3}{2}) = M_3(\frac{3}{2},\frac{3}{2})$ (see Tables 3.2.7 and 3.2.8 respectively). The sets of exceptional slopes and fillings of $M_3(-1,-2)$ and $M_3(-1,4)$ can be found in [**MP**], and we include them in Tables 3.2.7 and 3.2.8 respectively for completeness.

Remark Using the same type of arguments as in Section 3.1 it is easy to see that $M_3(-1,4) = M_3(4,\frac{3}{2}) = M_3(\frac{3}{2},\frac{3}{2})$.

Recall that $\Delta_0(T^H,T)$ is known to be 3 or 4.

Corollary 3.2.5. *No exceptional* (T^H,T) *pair* $(M_3(f);\alpha,\beta)$ *with* $\Delta(\alpha,\beta) = 4$ *exists, and all exceptional* (T^H,T) *pairs at distance* 3 *are described in Table 3.2.9.*

α	$M_3(-1,-2)(\alpha)$
∞	S^3
1	$T_{\begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}}$
-3, 5	$\big(D,(2,1),(2,1)\big)\bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}\big(D,(2,1),(3,1)\big)$
0, 2	$\big(S^2,(2,1),(3,1),(7,1),-1\big)$
-2, 4	$\big(S^2,(3,1),(3,1),(4,1),-1\big)$
-1, 3	$\big(S^2,(2,1),(4,1),(5,1),-1\big)$

TABLE 3.2.7. The set of exceptional fillings of the figure-8 knot $M_3(-1,-2)$.

α	$M_3(-1,4)(\alpha)$
0, ∞	$L(5,1)$
1	$L(10,3)$
3, $\frac{1}{3}$	$\big(D,(2,1),(2,1)\big)\bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}\big(D,(0,1),(-1,-1)\big)$
-1	$T_{\begin{pmatrix} -3 & 1 \\ -1 & 0 \end{pmatrix}}$
2, $\frac{1}{2}$	$\big(S^2,(2,1),(3,2),(3,2),-1\big)$

TABLE 3.2.8. The set of exceptional fillings of the figure-8 knot sister $M_3(-1,4)$.

Space	$M_3(1-\frac{1}{n},1-\frac{1}{m})$	$M_3(2-\frac{1}{n},-1)$	$M_3(3-\frac{1}{n},\frac{1}{2})$	$M_3(4,\frac{2}{3})$
(T^H,T)-pair	(3, 0)	(2, -1)	(1, 4)	(1, 4)
Restrictions	$n,m\notin\{0,\pm1\}$	$n\notin\{0,\pm1\}$	$n\notin\{0,\pm1\}$	-

TABLE 3.2.9. Every $(M_3(f);\alpha,\beta)$ exceptional (T^H,T) pair such that $\Delta(\alpha,\beta)=3$.

PROOF. We know from Table 3.2.1 that an exceptional slope α on $M_3(f)$ of type T^H lies in $\{\infty, 3, 2, 1, 0\}$. Every exceptional slope α is contained in some isolated exceptional filling instruction for M_3, and those slopes at distance 4 to some element of $\{\infty, 3, 2, 1, 0\}$ are easily enumerated using Tables 3.0.1-3.0.5. Any pair of exceptional slopes α, β) on $M_3(f)$ corresponding to an exceptional (T^H, T)-pair with $\Delta(\alpha, \beta) = 4$ must be one of

$$(3, -1), (2, \tfrac{2}{3}), (2, -2), (1, 5), (1, -3), (1, \tfrac{7}{3}), (0, 4), (0, -4), (0, \tfrac{4}{3}).$$

The restrictions imposed on $M_3(f)$ for α to correspond to a lens space filling dismiss all proposed pairs; for example when 0 is an exceptional slope on $M_3(f)$ producing a lens space, then $f = (n, 4 - n - \frac{1}{m})$ which rules out 4, -4, $\frac{4}{3}$ as being exceptional slopes for all choices of n and m (see Tables 3.0.1-3.0.5). Similarly, all possible distance 3 fillings are enumerated and examined. The result is Table 3.2.9 and Corollary 3.2.5. □

We collect Proposition 3.2.1-Corollary 3.2.5 in the following theorem.

THEOREM 3.2.6. *Table 3.2.10 shows the maximum $\Delta(\mathcal{C}_1, \mathcal{C}_2)$ obtained for exceptional pairs of type $(\mathcal{C}_1, \mathcal{C}_2)$ of the form $(M_3(f); \alpha, \beta)$ for $\mathcal{C} \in \{S^H, S, T^H, T\}$ and $M_3(f)$ with 1 boundary component. Table 3.2.11 shows the maximum $\Delta(\mathcal{C}_1, \mathcal{C}_2)$ obtained for exceptional pairs $(M_3(f); \alpha, \beta)$ of type $(\mathcal{C}_1, \mathcal{C}_2)$ for $\mathcal{C} \in \{S^H, S, T^H, T\}$ when $M_3(f)$ is a hyperbolic knot exterior in S^3. Numbers in boldface are $\Delta_0(\mathcal{C}_1, \mathcal{C}_2)$, and the entries include references to the complete description of all $(M_3(f); \alpha, \beta)$ realising the quoted maxima.*

	S^H	S	T^H	T
S^H	**0**	(conj.) $-\infty$, see Prop. 3.2.4	**1**, see Table 3.2.3	**2**, see Table 3.2.6
S		**0**, see Prop. 3.2.4	**1**, see Prop. 3.2.4	**3**, see Prop. 3.2.4
T^H			**1**, see Cor. 3.2.2	(conj.) 3, see Table 3.2.9
T				**8**, see Table 3.2.7

TABLE 3.2.10. Maximal $\Delta(\alpha, \beta)$ for $(M_3(f); \alpha, \beta)$ and descriptions of all exceptional pairs.

	S	T^H	T
S	$-\infty$, see Prop. 3.2.4	$-\infty$, see Prop. 3.2.4	$-\infty$, see Prop. 3.2.4
T^H		**1**, see Table 3.2.5	(conj.) 3, see Prop. 3.2.3
T			**8**, see Table 3.2.7

TABLE 3.2.11. Maximal $\Delta(\alpha, \beta)$ for $(M_3(f); \alpha, \beta)$ when $M_3(f)$ is a knot complement in S^3 and descriptions of all exceptional pairs.

Type-Z fillings. We complement Theorem 3.2.6 by outlining the corresponding result when one of the exceptional classes is of type Z, the class of Seifert spaces fibering over the sphere with exactly three exceptional fibres.

THEOREM 3.2.7. (Martelli-Petronio) *For $\mathcal{C} \in \{T, Z, T^H, S^H\}$, the best lower bounds for $\Delta_0(Z, \mathcal{C})$ obtained by pairs of the form $\{(M_3(f); \alpha_1, \alpha_2)\}$ of type $(Z, \mathcal{C})$ are given in Table 3.2.12.*

Type	T	Z	T^H	S	S^H
Z	7	6	2	2	1

TABLE 3.2.12. Lower bounds for $\Delta_0(\alpha, \beta)$ obtained from $\{(M_3(f); \alpha, \beta)\}$ of type $(Z, \mathcal{C})$.

Arguing as in Sections 3.2-3.2 and using the Appendix of [**MP**] easily allows us to complement Theorem 3.2.7 with a description of all $(M_3(f); \alpha, \beta)$ pairs realising the distances of Table 3.2.12.

PROPOSITION 3.2.8. *Every exceptional pair of the form $(M_3(f); \alpha_1, \alpha_2)$ with $M_3(f)$ having 1 boundary component that attains the bounds of Theorem 3.2.7 has f in $\{(-1, -2), (-1, 4), (4, \frac{3}{2}), (\frac{3}{2}, \frac{3}{2})\}$ or is described by one of the following:*

- $\big(M_3(f); Z, T^H\big)$;
  - $(M_3(n, \frac{p}{q}); 0, 2)$, where $n, \frac{p}{q} \in \mathbb{Z}\backslash\{0, 1, 2, 3\}$ with $\frac{p}{q} \neq 4 - n + \frac{1}{m}$ or $2 - \frac{1}{k}$, and $(n, \frac{p}{q}) \neq (4, \frac{1}{2}), (-1, -1)$,
  - $(M_3(n, 4 - n + \frac{1}{m}); 2, 0)$, where $n \in \mathbb{Z}\backslash\{0, 1, 2, 3\}$, $m \in \mathbb{Z}\backslash\{0\}$, and $(n, m) \neq (4, 2), (4, 1), (5, 1)$
  - $(M_3(\frac{n-1}{n}, \frac{m-1}{m}); 1, 3)$, where $n, m \in \mathbb{Z}\backslash\{0, \pm 1\}$,
  - $(M_3(\frac{n-1}{n}, \frac{3m-1}{m}); 3, 1)$, where $n \in \mathbb{Z}\backslash\{0, \pm 1\}$, $m \in \mathbb{Z}\backslash\{0, 1\}$, and $(n, m) \neq (2, -1)$.

- $\big(M_3(f); Z, S\big)$;
  - $(M_3(n, 4-n); 2, 0)$, where $n \in \mathbb{Z} \backslash \{0,1,2,3,4\}$,
- $\big(M_3(f); Z, S^H\big)$;
  - $(M_3(n, \frac{k}{nk+\epsilon}); 0, \infty)$, where $n \notin \{0,1,2,3\}, k \neq -1$,
  - $(M_3(\frac{p}{q}, \frac{r}{s}); 1, \infty)$, where $|pr-qs| = 1$, $\frac{p}{q}, \frac{r}{s} \notin \{0,1,2,3,3-\frac{1}{n}\}$, $\{\frac{p}{q}, \frac{r}{s}\} \neq \{-1,-1\}$, $\{\frac{3}{2}, \frac{5}{2}\}$, $\{4, \frac{1}{2}\}$,
  - $(M_3(\frac{p}{q}, \frac{r}{s}); 2, \infty)$, where $|pr-qs| = 1$ with $\frac{p}{q}, \frac{r}{s} \notin \{0,1,2,3,2-\frac{1}{n}\}$, $\{\frac{p}{q}, \frac{r}{s}\} \neq \{-1,-1\}$, $\{\frac{3}{2}, \frac{5}{2}\}$, $\{4, \frac{1}{2}\}$,
  - $(M_3(\frac{n-1}{n}, \frac{kn-\epsilon}{k(n-1)-\epsilon}); 3, \infty)$, where $n \notin \{0,1,2,3\}$, $k \neq 0$, and $(n,k) \neq (-1,\epsilon)$, $(\epsilon-1,-1)$

  where $n, k \in \mathbb{Z}$, $\epsilon = \pm 1$.

3.3. A class of toroidal manifold

Here we restrict our attention to a class of toroidal manifold appearing as a filling along a slope α on a hyperbolic manifold of the form $M_3(f)$ for only three pairs (f, α).

The JSJ decomposition theorem says that an irreducible graph manifold M containing only toroidal boundary components possesses a set of essential tori $\mathcal{E}$, unique up to isotopy, such that $M \backslash \mathcal{E} = \bigcup S_i$ where each S_i is a Seifert space. Moreover, the fibration of each S_i is unique up to isotopy provided we insist that any $S_i \cong \big(D, (2,1), (2,1)\big)$ is given the fibration of $\big(D, (2,1), (2,1)\big)$ (see [**FM**]).

Let S_1 and S_2 be Seifert spaces and $g : B_1 \to B_2$ be a homeomorphism between boundary components B_1 of S_1 and B_2 of S_2. We define the *intersection index* of $S_1 \cup_g S_2$ to be $\Delta(g(f_1), f_2)$ where f_1 is a fibre of B_1 and f_2 is a fibre of B_2.

The remarks above imply that the following definition is well-defined.

Definition Let M be an irreducible graph manifold containing only toroidal boundary components. We define the *intersection index of* M to be the maximum intersection index between pairs of Seifert spaces in the JSJ decomposition of M sharing a common boundary in M.

Remark If a map between boundary components of Seifert spaces S_1 and S_2 is described by a matrix then $S_1 \bigcup_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} S_2$ has intersection index equal to

$$\Delta\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = |b|.$$

Define $\mathcal{I}$ to be the class of irreducible graph manifolds with intersection index greater than 1. All examples of $(M_3(f);\alpha)$ with $M_3(f)(\alpha) \in \mathcal{I}$ are given in Table 3.3.1 along with the exceptional fillings of $M_3(f)$. We are unaware of this class of manifold being considered in the literature before now. Our interest in $\mathcal{I}$ comes from the rarity with which exceptional fillings in this class appear. We will return to the class $\mathcal{I}$ in Section 4.4.

$M_3(-1,-3)$	
$3 : \big(D,(2,1),(2,1)\big) \bigcup_{\begin{pmatrix}0&1\\1&0\end{pmatrix}} \big(D,(2,1),(4,1)\big)$	$2 : \big(S^2,(3,-2),(3,1),(5,1)\big)$
$1 : \big(S^2,(2,-1),(4,1),(6,1)\big)$	$0 : \big(S^2,(2,-1),(3,1),(8,1)\big)$
$-1 : T/_{\begin{pmatrix}4&1\\-1&0\end{pmatrix}}$	$-2 : \big(S^2,(2,-1),(4,1),(5,1)\big)$
$-3 : \big(D,(2,1),(2,1)\big) \bigcup_{\begin{pmatrix}1&2\\0&-1\end{pmatrix}} \big(D,(2,1),(3,1)\big)$	$\infty : L(2,1)$
$M_3(-3,-3)$	
$3 : \big(D,(2,1),(4,1)\big) \bigcup_{\begin{pmatrix}0&1\\1&0\end{pmatrix}} \big(D,(2,1),(4,1)\big)$	$2 : \big(S^2,(3,2),(5,-1),(5,-1)\big)$
$1 : \big(S^2,(2,1),(6,-1),(6,-1)\big)$	$0 : \big(S^2,(2,-1),(3,1),(10,1)\big)$
$-1 : \big(D,(2,1),(2,1)\big) \bigcup_{\begin{pmatrix}1&2\\0&-1\end{pmatrix}} \big(D,(2,1),(3,1)\big)$	$\infty : L(8,3)$
$M_3(-2,-2)$	
$3 : \big(D,(2,1),(3,1)\big) \bigcup_{\begin{pmatrix}0&1\\1&0\end{pmatrix}} \big(D,(2,1),(3,1)\big)$	$2 : \big(S^2,(3,-2),(4,1),(4,1)\big)$
$1 : \big(S^2,(2,-1),(5,1),(5,1)\big)$	$0 : \big(S^2,(2,-1),(3,1),(8,1)\big)$
$-1 : \big(S^2,(2,-1),(3,1),(7,1)\big)$	$\infty : L(3,1)$
$-2 : \big(D,(2,1),(2,1)\big) \bigcup_{\begin{pmatrix}2&3\\-1&-2\end{pmatrix}} \big(D,(2,1),(3,1)\big)$	

TABLE 3.3.1. Exceptional fillings of all $M_3(f)$ that admit a filling in $\mathcal{I}$.

CHAPTER 4

Consequences and questions

This Chapter is about natural research problems that arise from the results in Chapters 1-3. The problems range from the tractable and immediate described in Section 4.2 and Section 4.3 to the more involved and longer-term project outlined in Section 4.4.

Chapter 3 deals with the question from the Gordon program of enumerating all exceptional pairs $(M_3(f);\alpha_1,\alpha_2)$ with $\Delta_0(\alpha_1,\alpha_2) = \Delta_0(\mathcal{C}_1,\mathcal{C}_2)$ for $\mathcal{C}_1,\mathcal{C}_2 \in \{S^H, S, T^H, T, Z\}$ when $M_3(f)$ is a one-cusped hyperbolic manifold. In Sections 4.1 and 4.2 we will do the same for pairs coming from M_5 but not from M_3, namely we examine the problem of enumerating all $(M_5(f),\alpha_1,\alpha_2)$ of type $(\mathcal{C}_1,\mathcal{C}_2)$ with f not factoring through M_3 and $\Delta(\alpha,\beta) = \Delta_0(\mathcal{C}_i,\mathcal{C}_j)$. The specific case when $(\mathcal{C}_i,\mathcal{C}_j) = (S^H, T^H)$ is considered in Section 4.3.

In Section 4.4 we return to the class $\mathcal{I}$ introduced in Section 3.3. In particular we describe all one-cusped hyperbolic manifolds of the form $M_5(f)$ with a filling in $\mathcal{I}$ and describe $E(M_5(f))$ and the corresponding fillings for all such $M_5(f)$'s. These examples are then used to set out a longer-term research program.

4.1. Exceptional filling types

In this section we will give conditions on (f,α) that determine the type of $M_5(f)(\alpha)$ when f is a hyperbolic filling instruction on M_5 with four exceptional slopes not factoring through M_3, and α is an exceptional slope on $M_5(f)$. These conditions will be used when we examine the problem of describing all $(M_5(f);\alpha_1,\alpha_2)$ of type $(\mathcal{C}_1,\mathcal{C}_2)$ with $\Delta(\alpha_1,\alpha_2) = \Delta_0(\mathcal{C}_1,\mathcal{C}_2)$ in Sections 4.2 and 4.3.

Corollaries 1.3.1 and 1.3.2 say that, up to equivalence, a hyperbolic filling instruction f on M_5 either factors through $M_5(-1)$ and has $E(M_5(f)) = \{-1,0,1,\infty\}$, or does not factor through $M_5(-1)$ and has $E(M_5(f)) = \{0,1,\infty\}$, or is found in Tables 1.3.1-1.3.3. Up to (1.1)-(1.13), Tables 1.3.1 and 1.3.2 describe every filling instruction f not factoring through $M_5(-1)$ with $e(M_5(f)) > 3$, and Table 1.3.3 describes every filling instruction f factoring through $M_5(-1)$ but not factoring through M_3 with $e(M_5(f)) > 4$. We now begin the program of writing general conditions for the class of $M_3(f)(\alpha)$ when f is not found in Tables 1.3.1-1.3.3. To do so we will first establishing conditions on filling types of the exterior F of MT4CL introduced in Figure 0.0.2.

LEMMA 4.1.1. *Let* $f = (\frac{i_0}{j_0}, \frac{i_1}{j_1}, \frac{i_2}{j_2}, \frac{i_3}{j_3})$ *be a filling instruction on* F *with all* $\frac{i_k}{j_k}$ *non-empty. Up to the* D_8 *action on subscripts generated by* $\{(13), (0123)\}$*, the integers* $i_0, \ldots, j_3$ *satisfy exactly one of the Toroidal/Reducible/Type-Z/Cyclic conditions below and* $F(f)$ *is of that type:*

Toroidal conditions: *If all* $\frac{i_k}{j_k}$ *have* $|i_k| > 1$ *then*

$$F(f) = \big(D, (i_0, j_0), (i_2, j_2)\big) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \big(D, (i_1, j_1), (i_3, j_3)\big);$$

Reducible conditions: *If* $i_0 = 0$, $|i_2| \neq 1$, $|i_1 j_3 + i_3 j_1| \neq 1$, *or if* $i_0 = i_2 = 1$, $j_0 + j_2 = 0$, $i_1, i_3 \neq 1$ *then one respectively has*

$$F(f) = L(i_2, j_2) \# L(i_1 j_3 + i_3 j_1, i_1 j_3' + i_3' j_1),$$

$$F(f) = L(i_1, j_1) \# L(i_3, j_3);$$

Type-Z conditions: *If* $i_0 = 1$, $i_2, j_1, j_3 \neq 0$, $|j_2 + j_0 i_2|$, $|i_1|$, $|i_3| > 1$ *then*

$$\big(S^2, (j_2 + i_2 j_0, -i_2), (i_1, j_1), (i_3, j_3)\big)$$

Cyclic conditions: *See Table 4.1.1.*

Condition	*F(f)*
$i_0 = j_3 = 1, j_2 + i_2 j_0 = 0$	$L(i_1, j_1)$
$i_0 = j_2 + i_2 j_0 = 1$	$L(i_3(j_1 - i_1 i_2) + j_3 i_1, \nu i_1 + \mu(j_1 - i_1 i_2))$ $\nu i_3 - \mu j_3 = 1$
$i_0 = 1, j_2 + i_2 j_0 = -1$	$L(i_3(j_1 + i_1 i_2) + j_3 i_1, \nu i_1 + \mu(j_1 + i_1 i_2))$ $\nu i_3 - \mu j_3 = 1$
$i_0 = i_1 = 1$	$L((j_2 + i_2 j_0)(j_3 + j_1 i_3) - i_2 i_3, (j_2 + i_2 j_0)\nu - i_2 \mu)$ $\nu i_3 - \mu(j_3 + j_1 i_3) = 1$

TABLE 4.1.1. All f with cyclic $F(f)$.

PROOF. The conditions are easily seen to be mutually exclusive and exhaustive of all possibilities for f. The class of $F(f)$ is easily recognised using Proposition 1.2.3. $\square$

Remark We stress the important fact that the conditions of Lemma 4.1.1 are up to a D_8 action on the set of subscripts on slopes.

We will now see that Lemma 4.1.1 easily implies necessary and sufficient conditions for the filling type of exceptional -1, 1, 0, ∞ slopes on $M_5(f)$.

Corollary 4.1.2. *Let $f = (\frac{i_2-j_2}{i_2}, \frac{j_1}{j_1-i_1}, \frac{i_3+j_3}{j_3}, -\frac{j_0}{i_0})$, $(\frac{i_0}{j_0}+1, \frac{i_1}{j_1}, \frac{i_2}{j_2}, \frac{i_3}{j_3}+1)$, and $(-\frac{i_2}{j_2}, \frac{j_0}{i_0}, \frac{j_3}{i_3}, -\frac{i_1}{j_1})$ be hyperbolic filling instructions on M_5. Then 0, 1, ∞ (respectively) are exceptional toroidal/reducible/Type-Z/cyclic slopes on $M_5(f)$ if and only if $i_0, \ldots j_3$ satisfy the toroidal/reducible/Type-Z/cyclic conditions of Lemma 4.1.1.*

Proof. Let $f = (\frac{p}{q}, \frac{r}{s}, \frac{u}{v}, \frac{x}{y})$ be a hyperbolic filling instructions on M_5 in the statement, and α be 0, 1, or ∞. The identities from Theorem 1.1.3 are used to write $M_5(f)(\alpha)$ as $F(g)$. Setting $g = (\frac{i_0}{j_0}, \frac{i_1}{j_1}, \frac{i_2}{j_2}, \frac{i_3}{j_3})$ and solving for $p, \ldots, y$ we obtain the desired result using the Lemma 4.1.1. $\square$

We argue in the same way to get:

Corollary 4.1.3. *Let $f = (-1, \frac{i_2-j_2}{i_2}, \frac{i_3+2j_3}{j_3}, -\frac{j_0}{i_0})$ be a hyperbolic filling instruction on M_5. Then $M_5(f)(-1)$ is toroidal/reducible/Type-Z/cyclic type if and only if $i_0, \ldots, j_3$ satisfy the toroidal/reducible/ Type-Z/cyclic conditions of Lemma 4.1.1 with $\frac{i_1}{j_1} = \frac{2}{1}$.*

Now we complete the description of the class of each exceptional $M_5(f)(\alpha)$ when f is equivalent to a filling instruction contained in Tables 1.3.1-1.3.3.

Proposition 4.1.4. *For each f in Tables 1.3.1-1.3.3, $E(M_5(f))$ and the $M_5(f)(\alpha)$ type for each $\alpha \in E(M_5(f))$ are shown in Tables 4.1.2-4.1.13.*

Proof. The proof is by brute force. All pairs $(f, \frac{p}{q})$ with f a filling instruction from Table 1.3.3 and $\frac{p}{q} \in E(M_5(f))$ are examined individually using Theorem 1.1.3 and the result is Tables 4.1.2-4.1.13. $\square$

Remark In Tables 4.1.2-4.1.13 we will say that a set of exceptional slopes $\{\alpha_1, \ldots, \alpha_k\}$ is of type $\{\mathcal{C}_1, \ldots, \mathcal{C}_k\}$ to mean that α_i is of type $\mathcal{C}_i$ for each i.

f	$E(M_5(f))$ and types
$(-1,-\frac{1}{3},4,\frac{2}{3})$	$\{3,2,-1,0,1,\infty\}$, $\{T,T,Z,Z,T^H\}$
$(-1,-3,-\frac{1}{2},-2)$	$\{-2,2,-1,0,1,\infty\}$, $\{T,T,Z,Z,Z,Z\}$
$(-1,-3,-2,-2)$	$\{-3,-2,-1,0,1,\infty\}$, $\{T,T,T^H,Z,T,T^H\}$
$(-1,3,\frac{2}{3},4)$	$\{-\frac{1}{2},-\frac{1}{3},-1,0,1,\infty\}$, $\{T,T,Z,T^H,T,Z\}$
$(-1,\frac{1}{3},\frac{4}{3},\frac{4}{3})$	$\{\frac{1}{2},\frac{1}{3},-1,0,1,\infty\}$, $\{T,T,T,T^H,Z,Z\}$
$(-1,-\frac{1}{3},\frac{5}{2},\frac{2}{3})$	$\{-2,-1,0,1,2,\infty\}$, $\{T,Z,Z,Z,T,Z\}$
$(-1,-4,-2,-3)$, $(-1,-5,-3,-2)$, $(-1,-3,-3,-4)$, $(-1,-4,-5,-2)$, $(-1,-3,-5,-3)$, $(-1,-7,-2,-2)$, $(-1,-3,-2,-6)$, $(-1,-\frac{5}{3},4,-2)$ $(-1,-3,4,-\frac{2}{3})$, $(-1,3,5,-\frac{1}{2})$	$E(M_5(f))=\{-2,-1,0,1,\infty\}$ types $\{T,Z,Z,T,T^H\}$
$(-1,-4,-4,-2)$, $(-1,-6,-2,-2)$, $(-1,-3,-3,-3)$, $(-1,-5,-2,-2)$, $(-1,-3,-2,-4)$, $(-1,-3,-2,-3)$, $(-1,-4,-2,-2)$	$E(M_5(f))=\{-2,-1,0,1,\infty\}$ types $\{Z,Z,Z,T,T^H\}$
$(-1,-3,-2,-5)$	$\{-2,-1,0,1,\infty\}$, $\{Z,T^H,Z,T,T^H\}$
$(-1,-5,-3,-2)$	$\{-2,-1,0,1,\infty\}$, $\{T,T^H,Z,T,T^H\}$
$(-1,-3,-2,-5)$	$\{-2,-1,0,1,\infty\}$, $\{T,Z,T^H,Z,Z\}$

TABLE 4.1.2. Exceptional sets for $M_5(f)$ for f in Table 1.3.3, part 1/4.

$f = (-1, \frac{p}{q}, -2, \frac{1}{3}),\ \frac{p}{q} \in \mathbb{Q}\backslash\{-2, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, 2\},\ E(M_5(f)) = \{2, -1, 0, 1, \infty\}$
$\text{types} \begin{cases} \{T^H, T, Z, T, T^H\} \text{ if } \frac{p}{q} = \frac{k+1}{2k} \neq \frac{1}{4}; \\ \{T^H, T, Z, Z, T^H\} \text{ if } \frac{p}{q} = \frac{1}{4}; \\ \{Z, Z, Z, T, T^H\} \text{ if } \lvert q \rvert = 1; \\ \{Z, T, T^H, T, T^H\} \text{ if } \frac{p}{q} = 1 + \frac{1}{k}; \\ \{Z, T, Z, Z, T^H\} \text{ if } \frac{p}{q} = \frac{1}{k}; \\ \{Z, T, Z, T, S^H\} \text{ if } \lvert 5q + 5p \rvert = 1; \\ \{Z, T, Z, T, T^H\} \text{ otherwise.} \end{cases}$
$f = (-1, -\frac{2}{3}, -2, \frac{u}{v}),\ \frac{u}{v} \in \mathbb{Q}\backslash\{-1, 0, \frac{1}{2}, 1, \frac{3}{2}, 2, 3\},\ E(M_5(f)) = \{2, -1, 0, 1, \infty\}$
$\text{types} \begin{cases} \{Z, T, T^H, T, T^H\} \text{ if } \lvert u - v \rvert = 1; \\ \{T, Z, T^H, T, T^H\} \text{ if } \lvert v \rvert = 1; \\ \{T, T, S^H, T, T^H\} \text{ if } 3v - 16u = \pm 1 - 8; \\ \{T, T, T^H, Z, T^H\} \text{ if } \frac{u}{v} = 1 + \frac{1}{n}; \\ \{T, T, T^H, T, S^H\} \text{ if } \lvert 5u + v \rvert = 1; \\ \{T, T, T^H, T, T^H\} \text{ otherwise.} \end{cases}$
$f = (-1, -\frac{1}{3}, \frac{r}{s}, \frac{2}{3}),\ \frac{r}{s} \in \mathbb{Q}\backslash\{-1, 0, \frac{1}{2}, 1, \frac{3}{2}, 2, 3\},\ E(M_5(f)) = \{2, -1, 0, 1, \infty\}$
$\text{types} \begin{cases} \{T, Z, Z, Z, Z\} \text{ if } \frac{r}{s} = \frac{2k-1}{k-1}; \\ \{T, T, T^H, Z, Z\} \text{ if } \lvert r - s \rvert = 1; \\ \{T, T, Z, T^H, Z\} \text{ if } \lvert r \rvert = 1; \\ \{T, T, Z, Z, T^H\} \text{ if } \lvert s \rvert = 1; \\ \{T, T, Z, Z, Z\} \text{ otherwise.} \end{cases}$

Table 4.1.3. Exceptional sets of $M_5(f)$ for f in Table 1.3.3, part 2/4.

$f=(-1,-3,\frac{r}{s},-2)$, $\frac{r}{s}\in\mathbb{Q}\backslash\{-1,0,\frac{1}{2},1,\frac{3}{2},2,3\}$, $E(M_5(f))=\{-2,-1,0,1,\infty\}$
$\text{types}\begin{cases}\{T,Z,T^H,T,Z\}\text{ if }\frac{r}{s}=\pm1+\frac{1}{k};\\ \{T,Z,Z,Z,Z\}\text{ if }\frac{r}{s}=\frac{1}{k};\\ \{T,Z,Z,T,T^H\}\text{ if }\frac{r}{s}\in\mathbb{Z}\backslash\{-2\};\\ \{T,Z,Z,T,Z\}\text{ otherwise.}\end{cases}$
$f=(-1,-\frac{3}{2},4,\frac{u}{v})$, $\frac{u}{v}\in\mathbb{Q}\backslash\{-1,0,\frac{1}{2},1,\frac{3}{2},2,3\}$, $E(M_5(f))=\{-2,-1,0,1,\infty\}$
$\text{types}\begin{cases}\{Z,Z,T^H,T,T^H\}\text{ if }\lvert v\rvert=1;\\ \{T,T,Z,Z,T^H\}\text{ if }\frac{u}{v}=\frac{k+1}{k};\\ \{T,T,Z,Z,S^H\}\text{ if }\lvert v-6u\rvert=1;\\ \{T,T,Z,Z,T^H\}\text{ otherwise.}\end{cases}$
$f=(-1,\frac{p}{q},4,-\frac{1}{2})$, $\frac{p}{q}\in\mathbb{Q}\backslash\{-2,-1,-\frac{1}{2},0,\frac{1}{2},1,2\}$, $E(M_5(f))=\{-2,-1,0,1,\infty\}$
$\text{types}\begin{cases}\{Z,Z,Z,T,T^H\}\text{ if }\lvert q\rvert=1;\\ \{T,T,T^H,T,T^H\}\text{ if }\frac{p}{q}=\frac{k+1}{k};\\ \{T,T,Z,Z,T^H\}\text{ if }\lvert p\rvert=1;\\ \{T,T,Z,T,S^H\}\text{ if }\lvert 6p+5q\rvert=1;\\ \{T,T,Z,T,T^H\}\text{ otherwise.}\end{cases}$

TABLE 4.1.4. Exceptional sets for $M_5(f)$ for f in Table 1.3.3, part 3/4.

f	$E(M_5(f))$ and types
$(-1,-\frac{1}{4},4,\frac{3}{4})$, $(-1,-\frac{1}{5},5,\frac{2}{3})$, $(-1,-\frac{1}{3},5,\frac{4}{5})$, $(-1,-\frac{1}{7},4,\frac{2}{3})$, $(-1,-\frac{1}{3},7,\frac{3}{4})$, $(-1,-\frac{1}{3},4,\frac{6}{7})$, $(-1,-\frac{1}{4},7,\frac{2}{3})$, $(-1,-\frac{1}{3},-2,\frac{2}{5})$ $(-1,\frac{1}{3},-3,\frac{1}{3})$, $(-1,-\frac{2}{3},-3,\frac{4}{3})$	$E(M_5(f))=\{2,-1,0,1,\infty\}$ types $\{T,T,Z,Z,T^H\}$
$(-1,-\frac{1}{4},6,\frac{2}{3})$, $(-1,-\frac{1}{3},6,\frac{3}{4})$, $(-1,-\frac{1}{4},5,\frac{2}{3})$, $(-1,-\frac{1}{3},4,\frac{4}{5})$, $(-1,-\frac{1}{3},5,\frac{3}{4})$,	$E(M_5(f))=\{-2,-1,0,1,\infty\}$ types $\{Z,T,Z,Z,T^H\}$
$(-1,-\frac{3}{5},-2,\frac{2}{3})$	$\{2,-1,0,1,\infty\}$, $\{T,Z,Z,Z,T^H\}$

Table 4.1.5. Exceptional sets for $M_5(f)$ for f in Table 1.3.3, part 4/4.

f	$E(M_5(f))$ and types
$(-2,-\frac{1}{3},3,\frac{2}{3})$, $(-2,-\frac{2}{3},-2,\frac{2}{3})$, $(-2,\frac{1}{3},-3,\frac{1}{3})$, $(-2,-\frac{1}{3},-2,\frac{2}{5})$, $(-2,-3,-\frac{1}{2},-2)$, $(-2,-\frac{1}{2},-\frac{3}{2},\frac{1}{3})$, $(-2,-\frac{1}{2},-3,\frac{1}{3})$, $(-2,-\frac{1}{2},-3,\frac{3}{5})$	$E(M_5(f))=\{2,0,1,\infty\}$ types $\{T,Z,Z,Z\}$
$(-2,-2,-\frac{1}{3},3)$	$E(M_5(f))=\{2,0,1,\infty\}$, $\{T,S,Z,Z\}$
$(-2,-2,\frac{1}{4},3)$	$E(M_5(f))=\{\frac{1}{2},0,1,\infty\}$, $\{Z,S,Z,Z\}$
$(-2,\frac{2}{5},\frac{3}{4},\frac{3}{2})$	$E(M_5(f))=\{\frac{1}{2},0,1,\infty\}$, $\{T^H,T^H,Z,T\}$
$(-2,\frac{1}{5},\frac{4}{3},\frac{3}{2})$, $(-2,\frac{1}{5},\frac{3}{2},\frac{3}{2})$, $(-2,\frac{1}{6},\frac{3}{2},\frac{3}{2})$, $(-2,\frac{1}{7},\frac{3}{2},\frac{3}{2})$, $(-2,\frac{1}{3},\frac{2}{3},\frac{5}{3})$, $(-2,\frac{1}{3},\frac{2}{3},\frac{5}{3})$	$E(M_5(f))=\{\frac{1}{2},0,1,\infty\}$, types $\{Z,T^H,Z,T\}$
$(-2,\frac{1}{4},\frac{2}{3},\frac{5}{3})$, $(-2,\frac{1}{4},\frac{3}{2},\frac{4}{3})$, $(-2,\frac{1}{3},\frac{3}{2},\frac{4}{3})$, $(-2,\frac{1}{8},\frac{3}{2},\frac{3}{2})$, $(-2,\frac{1}{5},\frac{6}{5},\frac{3}{2})$, $(-2,\frac{3}{8},\frac{3}{4},\frac{3}{2})$, $(-2,\frac{2}{3},\frac{3}{4},\frac{2}{3})$, $(-2,\frac{2}{3},\frac{1}{3},3)$,	$E(M_5(f))=\{\frac{1}{2},0,1,\infty\}$, types $\{T,T^H,Z,T\}$
$(-2,3,\frac{1}{3},4)$	$E(M_5(f))=\{\frac{1}{2},0,1,\infty\}$, $\{T,T^H,Z,Z\}$
$(-2,-2,\frac{1}{5},3)$	$E(M_5(f))=\{\frac{1}{2},0,1,\infty\}$, $\{T,S,Z,Z\}$
$(-2,\frac{2}{3},\frac{3}{5},\frac{3}{2})$	$E(M_5(f))=\{\frac{1}{2},0,1,\infty\}$, $\{T^H,T^H,Z,Z\}$
$(-2,\frac{1}{3},\frac{1}{3},3)$	$E(M_5(f))=\{\frac{1}{2},0,1,\infty\}$, $\{T,S,T^H,T\}$
$(-2,\frac{3}{5},\frac{2}{3},\frac{4}{3})$	$E(M_5(f))=\{\frac{1}{2},0,1,\infty\}$, $\{T,T^H,Z,T^H\}$

TABLE 4.1.6. Exceptional sets for $M_5(f)$ for f in Table 1.3.2, part 1/2.

f	$E(M_5(f))$ and types
$(-2,4,5,-\frac{3}{2})$, $(-2,3,4,-\frac{4}{3})$, $(-2,-2,4,-\frac{5}{3})$, $(-2,-2,-3,-5)$, $(-2,-4,-3,-3)$, $(-2,-2,-2,-7)$, $(-2,-6,-2,-3)$, $(-2,-2,-5,-4)$, $(-2,-3,-5,-3)$, $(-2,-2,-3,-5)$, $(-2,-4,-3,-3)$, $(-2,-3,-2,-4)$, $(-2,-3,-5,-3)$, $(-2,-2,-2,-4)$,	$E(M_5(f)) = \{-1,0,1,\infty\}$, types $\{T,Z,T,T^H\}$
$(-2,-\frac{1}{2},5,3)$	$E(M_5(f)) = \{-1,0,1,\infty\}$, $\{T,S,Z,Z\}$
$(-2,3,\frac{3}{2},-\frac{1}{2})$	$E(M_5(f)) = \{-1,0,1,\infty\}$, $\{T,T^H,T,T^H\}$
$(-2,-\frac{2}{3},4,-3)$	$E(M_5(f)) = \{-1,0,1,\infty\}$, $\{T,Z,T,Z\}$
$(-2,3,\frac{1}{3},-3)$	$E(M_5(f)) = \{-1,0,1,\infty\}$, $\{T,Z,Z,Z\}$
$(-2,\frac{5}{3},\frac{3}{2},-\frac{1}{2})$, $(-2,\frac{3}{2},\frac{5}{3},-\frac{1}{3})$, $(-2,\frac{4}{3},\frac{7}{3},-\frac{1}{2})$	$E(M_5(f)) = \{-1,0,1,\infty\}$, $\{T,T^H,T,Z\}$
$(-2,\frac{2}{3},\frac{5}{2},-\frac{1}{3})$	$E(M_5(f)) = \{-1,0,1,\infty\}$, $\{T,T^H,Z,T\}$
$(-2,\frac{4}{3},\frac{3}{2},\frac{1}{3})$	$E(M_5(f)) = \{-1,0,1,\infty\}$, $\{T,T^H,Z,Z\}$
$(-2,-2,-2,-4)$, $(-2,-3,-2,-3)$, $(-2,-2,-2,-5)$, $(-2,-2,-2,-6)$, $(-2,-5,-2,-3)$, $(-2,-2,-4,-4)$, $(-2,-3,-4,-3)$, $(-2,-3,-3,-3)$, $(-2,-2,-3,-4)$,	$E(M_5(f)) = \{-1,0,1,\infty\}$, types $\{T,T^H,Z,Z\}$
$(-2,-4,-2,-3)$	$E(M_5(f)) = \{-1,0,1,\infty\}$, $\{Z,Z,T,Z\}$
$(-2,\frac{3}{2},\frac{5}{2},-\frac{2}{3})$	$E(M_5(f)) = \{-1,0,1,\infty\}$, $\{T,Z,T,T\}$

Table 4.1.7. Exceptional sets for $M_5(f)$ for f in Table 1.3.2, part 2/2.

$f=(-2,-\frac{1}{2},3,3)$, $E(M_5(f))=\{-1,-\frac{1}{2},0,1,\infty\}$, types $\{Z,T,S,Z,Z\}$
$f=(-2,\frac{3}{2},\frac{3}{2},-2)$, $E(M_5(f))=\{-1,-\frac{1}{2},0,1,\infty\}$, types $\{T,T,T^H,T,T\}$
$f=(-2,-3,-\frac{1}{2},-2)$, $E(M_5(f))=\{-1,2,0,1,\infty\}$, types $\{Z,T,Z,Z,Z\}$
$f=(-2,-\frac{1}{3},3,\frac{2}{3})$, $E(M_5(f))=\{-1,2,0,1,\infty\}$, types $\{Z,T,Z,Z,Z\}$
$f=(-2,-\frac{1}{2},3,\frac{2}{3})$, $E(M_5(f))=\{-1,2,0,1,\infty\}$, types $\{Z,Z,Z,Z,Z\}$
$f=(-2,-2,-2,-2)$, $E(M_5(f))=\{-2,-1,0,1,\infty\}$, types $\{T,Z,Z,T,T^H\}$
$f=(-2,\frac{1}{3},3,\frac{1}{3})$, $E(M_5(f))=\{-2,-1,0,1,\infty\}$, types $\{T,Z,Z,Z,S\}$
$f=(-2,\frac{1}{3},\frac{3}{2},\frac{3}{2})$, $E(M_5(f))=\{\frac{1}{3},\frac{1}{2},0,1,\infty\}$, types $\{T,Z,T^H,Z,T\}$
$f=(-2,-2,\frac{1}{3},3)$, $E(M_5(f))=\{\frac{1}{3},\frac{1}{2},0,1,\infty\}$, types $\{T,Z,S,Z,Z\}$
$f=(-2,-\frac{1}{2},-2,\frac{3}{2})$, $E(M_5(f))=\{\frac{3}{2},2,0,1,\infty\}$, types $\{T,Z,Z,S,Z\}$
$f=(-2,\frac{3}{2},\frac{3}{2},\frac{u}{v})$, $\frac{u}{v}\in\mathbb{Q}\backslash\{-1,0,\frac{1}{2},1,2\}$, $E(M_5(f))=\{-1,0,1,\infty\}$ $\text{types}\begin{cases}\{T^H,T^H,T,T\} \text{ if } \frac{u}{v}=4;\\ \{Z,T^H,T,T\} \text{ if } \frac{u}{v}=4+\frac{1}{n};\\ \{T,S^H,T,T\} \text{ if } \lvert 5u-14v\rvert=1;\\ \{T,T^H,Z,T\} \text{ if } \lvert u-v\rvert=1;\\ \{T,T^H,T,Z\} \text{ if } \lvert u\rvert=1;\\ \{T,T^H,T,T\} \text{ otherwise.}\end{cases}$
$f=(-2,\frac{p}{q},\frac{5}{2},-\frac{1}{2})$, $\frac{p}{q}\in\mathbb{Q}\backslash\{-1,0,\frac{1}{2},1,2\}$, $E(M_5(f))=\{-1,0,1,\infty\}$ $\text{types}\begin{cases}\{Z,T^H,T,Z\} \text{ if } \frac{p}{q}=1+\frac{1}{n}, n\neq\pm1;\\ \{T,Z,Z,Z\} \text{ if } \lvert p\rvert=1;\\ \{T,Z,T,T^H\} \text{ if } \lvert q\rvert=1;\\ \{T,Z,T,Z\} \text{ otherwise.}\end{cases}$
$f=(-2,\frac{3}{2},\frac{r}{s},-\frac{1}{2})$, $\frac{r}{s}\in\mathbb{Q}\backslash\{-1,0,\frac{1}{2},1,2\}$, $E(M_5(f))=\{-1,0,1,\infty\}$ $\text{types}\begin{cases}\{Z,T^H,T,T^H\} \text{ if } \frac{r}{s}=3;\\ \{Z,T^H,T,Z\} \text{ if } \frac{r}{s}=2+\frac{1}{n},\ n\neq1;\\ \{T,S^H,T,Z\} \text{ if } \lvert 19r-12s\rvert=1;\\ \{T,T^H,Z,Z\} \text{ if } \lvert r\rvert=1;\\ \{T,T^H,T,T^H\} \text{ if } \lvert s\rvert=1;\\ \{T,T^H,T,Z\} \text{ otherwise.}\end{cases}$

TABLE 4.1.8. Exceptional sets for $M_5(f)$ for f in Table 1.3.1, part 1/6.

$f=(-2,-2,\frac{r}{s},-3)$, $\frac{r}{s}\in\mathbb{Q}\backslash\{-1,0,\frac{1}{2},1,2\}$, $E(M_5(f))=\{-1,0,1,\infty\}$
$\text{types}\begin{cases}\{T,T^H,T,Z\} \text{ if } \lvert r-s\rvert=1;\\ \{T,Z,Z,Z\} \text{ if } \lvert r\rvert=1;\\ \{T,Z,T,T^H\} \text{ if } \lvert s\rvert=1;\\ \{T,Z,T,Z\} \text{ otherwise.}\end{cases}$
$f=(-2,-\frac{1}{2},4,\frac{u}{v})$, $\frac{u}{v}\in\mathbb{Q}\backslash\{-1,0,\frac{1}{2},1,2\}$, $E(M_5(f))=\{-1,0,1,\infty\}$
$\text{types}\begin{cases}\{Z,S,Z,Z\} \text{ if } \frac{u}{v}=3;\\ \{Z,T^H,Z,S\} \text{ if } \frac{u}{v}=4;\\ \{T,Z,S,Z\} \text{ if } \frac{u}{v}=\frac{3}{2};\\ \{Z,Z,Z,T^H\} \text{ if } \frac{u}{v}=5;\\ \{T,T^H,Z,T^H\} \text{ if } \frac{u}{v}=\frac{7}{2};\\ \{Z,Z,Z,Z\} \text{ if } \frac{u}{v}\in\mathbb{Z}\backslash\{-2,-3,3,2,4\};\\ \{T,T^H,Z,Z\} \text{ if } \lvert u-3v\rvert=1,\ \frac{u}{v}\notin\{\frac{7}{2},4\};\\ \{T,Z,T^H,Z\} \text{ if } \lvert 2u-3v\rvert=1;\\ \{T,Z,Z,T^H\} \text{ if } \lvert 4v-u\rvert=1, \frac{u}{v}\neq\frac{7}{2}\\ \{T,Z,Z,Z\} \text{ otherwise.}\end{cases}$
$f=(-2,\frac{p}{q},4,-\frac{3}{2})$, $\frac{p}{q}\in\mathbb{Q}\backslash\{-1,0,\frac{1}{2},1,2\}$, $E(M_5(f))=\{-1,0,1,\infty\}$
$\text{types}\begin{cases}\{Z,Z,T,T^H\} \text{ if } \lvert q\rvert=1;\\ \{T,T^H,T,Z\} \text{ if } \lvert p-q\rvert=1;\\ \{T,Z,Z,Z\} \text{ if } \lvert p\rvert=1;\\ \{T,Z,T,Z\} \text{ otherwise.}\end{cases}$
$f=(-2,3,\frac{r}{s},-\frac{3}{2})$, $\frac{r}{s}\in\mathbb{Q}\backslash\{-1,0,\frac{1}{2},1,2\}$, $E(M_5(f))=\{-1,0,1,\infty\}$
$\text{types}\begin{cases}\{T^H,T^H,T,Z\} \text{ if } \frac{r}{s}=\frac{3}{2};\\ \{T^H,Z,T,Z\} \text{ if } \frac{r}{s}=2+\frac{1}{n},\ n\neq 1;\\ \{Z,T^H,T,Z\} \text{ if } \lvert r-s\rvert=1;\\ \{Z,Z,Z,Z\} \text{ if } \lvert r\rvert=1;\\ \{Z,Z,T,T^H\} \text{ if } \lvert s\rvert=1;\\ \{Z,Z,T,Z\} \text{ otherwise.}\end{cases}$

Table 4.1.9. Exceptional sets for $M_5(f)$ for f in Table 1.3.1, part 2/6.

$f=(-2,\frac{p}{q},\frac{1}{3},3),\ \frac{p}{q}\in\mathbb{Q}\backslash\{0,1,-1,\frac{1}{2},2\},\ E(M_5(f))=\{\frac{1}{2},0,1,\infty\}$ $\text{types}\begin{cases}\{Z,S,Z,Z\} \text{ if } \lvert q\rvert=1;\\ \{T,T^H,Z,T\} \text{ if } \lvert q-p\rvert=1;\\ \{T,S,T^H,T\} \text{ if } \lvert p\rvert=1;\\ \{T,S,Z,T\} \text{ otherwise.}\end{cases}$
$f=(-2,\frac{1}{3},\frac{r}{s},\frac{3}{2}),\ \frac{p}{q}\in\mathbb{Q}\backslash\{0,1,-1,\frac{1}{2},2\},\ E(M_5(f))=\{\frac{1}{2},0,1,\infty\}$ $\{\frac{1}{2},0,1,\infty\}\ \text{types}\begin{cases}\{T^H,Z,T^H,T\} \text{ if } \lvert r\rvert=1;\\ \{Z,T^H,Z,T\} \text{ if } \lvert s-r\rvert=1;\\ \{Z,Z,Z,Z\} \text{ if } \lvert s\rvert=1;\\ \{Z,Z,T,T\} \text{ otherwise.}\end{cases}$
$f=(-2,\frac{1}{4},\frac{r}{s},\frac{3}{2}),\ \frac{p}{q}\in\mathbb{Q}\backslash\{0,1,-1,\frac{1}{2},2\},\ E(M_5(f))=\{\frac{1}{2},0,1,\infty\}$ $\text{types}\begin{cases}\{T,T^H,Z,T\} \text{ if } \lvert s-r\rvert=1;\\ \{T,Z,T^H,T\} \text{ if } \lvert r\rvert=1;\\ \{T,Z,Z,Z\} \text{ if } \lvert s\rvert=1;\\ \{T,Z,Z,T\} \text{ otherwise.}\end{cases}$
$f=(-2,\frac{2}{3},\frac{2}{3},\frac{u}{v}),\ \frac{p}{q}\in\mathbb{Q}\backslash\{0,1,-1,\frac{1}{2},2\},\ E(M_5(f))=\{\frac{1}{2},0,1,\infty\}$ $\text{types}\begin{cases}\{Z,T^H,Z,T\} \text{ if } \lvert u-v\rvert=1;\\ \{T,S^H,T,T\} \text{ if } \lvert 5u-16v\rvert=1;\\ \{T,T^H,T,Z\} \text{ if } \lvert u\rvert=1;\\ \{T,T^H,T,T\} \text{ otherwise.}\end{cases}$
$f=(-2,\frac{p}{q},\frac{2}{3},\frac{3}{2}),\ \frac{p}{q}\in\mathbb{Q}\backslash\{0,1,-1,\frac{1}{2},2\},\ E(M_5(f))=\{\frac{1}{2},0,1,\infty\}$ $\text{types}\begin{cases}\{Z,T^H,Z,T\} \text{ if } \lvert q\rvert=1;\\ \{T,S^H,Z,T\} \text{ if } \lvert 11q-8p\rvert=1;\\ \{T,T^H,T^H,T\} \text{ if } \lvert q+2p\rvert=1;\\ \{T,T^H,Z,Z\} \text{ if } p-q=-1,\ \text{ or } p-q=-2;\\ \{T,T^H,Z,T\} \text{ otherwise.}\end{cases}$

TABLE 4.1.10. Exceptional sets for $M_5(f)$ for f in Table 1.3.1, part 3/6.

$f=(-2,-2,-\frac{1}{2},\frac{u}{v}),\ \frac{u}{v}\in\mathbb{Q}\backslash\{0,1,-1,\frac{1}{2},2\},\ E(M_5(f))=\{2,0,1,\infty\}$
$\text{types}\begin{cases}\{Z,T^H,Z,Z\} \text{ if } \frac{u}{v}=4;\\ \{Z,S,Z,Z\} \text{ if } \frac{u}{v}=3;\\ \{Z,Z,Z,Z\} \text{ if } \lvert v\rvert=1;\\ \{T,T^H,Z,Z\} \text{ if } \lvert 3v-u\rvert=1;\\ \{T,Z,T^H,Z\} \text{ if } \lvert u-v\rvert=1;\\ \{T,Z,Z,T^H\} \text{ if } \lvert u\rvert=1;\\ \{T,Z,Z,Z\} \text{ otherwise.}\end{cases}$
$f=(-2,-\frac{1}{2},-2,\frac{u}{v}),\ \frac{u}{v}\in\mathbb{Q}\backslash\{0,1,-1,\frac{1}{2},2\},\ E(M_5(f))=\{2,0,1,\infty\}$
$\text{types}\begin{cases}\{Z,S,Z,Z\} \text{ if } \frac{u}{v}=3;\\ \{Z,Z,S,Z\} \text{ if } \frac{u}{v}=\frac{3}{2};\\ \{Z,Z,Z,S\} \text{ if } \frac{u}{v}=-\frac{1}{2};\\ \{T^H,Z,Z,Z\} \text{ if } \lvert 2u-v\rvert=1;\\ \{Z,T^H,Z,Z\} \text{ if } \lvert 3v-u\rvert=1;\\ \{Z,Z,T^H,Z\} \text{ if } \lvert 3v-2u\rvert=1;\\ \{Z,Z,Z,T^H\} \text{ if } \lvert v+2u\rvert=1;\\ \{Z,Z,Z,Z\} \text{ otherwise.}\end{cases}$
$f=(-2,\frac{p}{q},-2,\frac{1}{3}),\ \frac{u}{v}\in\mathbb{Q}\backslash\{0,1,-1,\frac{1}{2},2\},\ E(M_5(f))=\{2,0,1,\infty\}$
$\text{types}\begin{cases}\{T^H,Z,Z,Z\} \text{ if } \frac{p}{q}=-\frac{1}{2};\\ \{Z,Z,Z,Z\} \text{ if } \lvert p\rvert=1;\\ \{T,T^H,T,Z\} \text{ if } \lvert q-p\rvert=1;\\ \{T,Z,T,T^H\} \text{ if } \lvert q\rvert=1;\\ \{T,Z,T,Z\} \text{ otherwise.}\end{cases}$
$f=(-2,-\frac{1}{2},\frac{r}{s},\frac{2}{3}),\ \frac{u}{v}\in\mathbb{Q}\backslash\{0,1,-1,\frac{1}{2},2\},\ E(M_5(f))=\{2,0,1,\infty\}$
$\text{types}\begin{cases}\{T^H,Z,Z,Z\} \text{ if } \frac{r}{s}=-3;\\ \{Z,Z,Z,Z\} \text{ if } \lvert s\rvert=1;\\ \{T,T^H,Z,T\} \text{ if } \lvert s-r\rvert=1;\\ \{T,Z,T^H,T\} \text{ if } \lvert r\rvert=1;\\ \{T,Z,Z,T\} \text{ otherwise.}\end{cases}$

Table 4.1.11. Exceptional sets for $M_5(f)$ for f in Table 1.3.1, part 4/6.

$f=(-2,\frac{p}{q},3,\frac{u}{v})$, $E(M_5(f))=\{-1,0,1,\infty\}$, $\frac{p}{q},\frac{u}{v}\in\mathbb{Q}\backslash\{0,1,-1,\frac{1}{2},2\}$
-1 is of type $\begin{cases} T^H \text{ if } \frac{u}{v}=-\frac{1}{2} \text{ and } \lvert p\rvert=1; \\ Z \text{ if } \frac{u}{v}\neq-\frac{1}{2} \text{ and } \lvert p\rvert=1; \\ T \text{ if } \lvert p\rvert\neq 1. \end{cases}$
0 is of type $\begin{cases} T^H \text{ if } \frac{u}{v}=3 \text{ and } \lvert q-p\rvert=1; \\ S \text{ if } \frac{u}{v}=3 \text{ and } \lvert q-p\rvert\neq 1; \\ S^H \text{ if } \lvert 3v-u\rvert=1, \lvert 3p-q+2v(q-p)\rvert=1; \\ T^H \text{ if } \lvert 3v-u\rvert=1, \lvert 3p-q+2v(q-p)\rvert\neq 1; \\ S^H \text{ if } \lvert q-p\rvert=1, \lvert 3u-7v+2q(u-3v)\rvert=1; \\ T^H \text{ if } \lvert q-p\rvert=1, \lvert 3u-7v+2q(u-3v)\rvert\neq 1; \\ Z \text{ otherwise.} \end{cases}$
1 is of type $\begin{cases} S \text{ if } v(1-q)+uq=0 \text{ and } \lvert p\rvert=1; \\ T^H \text{ if } \lvert v(1-q)+uq\rvert=1 \text{ and } \lvert p\rvert=1; \\ Z \text{ if } \lvert v(1-q)+uq\rvert>1 \text{ and } \lvert p\rvert=1; \\ S \text{ if } \lvert u-v\rvert=1 \text{ and } q+pv=0; \\ T^H \text{ if } \lvert u-v\rvert=1 \text{ and } \lvert q+pv\rvert=1; \\ Z \text{ if } \lvert u-v\rvert=1 \text{ and } \lvert q+pv\rvert>1; \\ S \text{ if } \frac{p}{q}=\frac{v-u}{v}=\frac{1}{n}, \text{for some } n; \\ T \text{ otherwise.} \end{cases}$
∞ is of type $\begin{cases} T^H \text{ if } \frac{u}{v}=\frac{1}{3} \text{ and } \lvert q\rvert=1; \\ S \text{ if } \frac{u}{v}=\frac{1}{3} \text{ and } \lvert q\rvert\neq 1; \\ S^H \text{ if } \lvert 3u-v\rvert=1 \text{ and } \lvert(2u-1)q-2p\rvert=1; \\ T^H \text{ if } \lvert 3u-v\rvert=1 \text{ and } \lvert(2u-1)q-2p\rvert\neq 1; \\ S^H \text{ if } \lvert q\rvert=1, \lvert 6\epsilon u-7u+3v-2\epsilon v-6\epsilon up+2\epsilon vp\rvert=1, \epsilon=\pm 1; \\ T^H \text{ if } \lvert q\rvert=1, \lvert 6\epsilon u-7u+3v-2\epsilon v-6\epsilon up+2\epsilon vp\rvert\neq 1, \epsilon=\pm 1; \\ Z \text{ otherwise.} \end{cases}$

TABLE 4.1.12. Exceptional sets for $M_5(f)$ for f in Table 1.3.1, part 5/6.

$f=(-2,\frac{p}{q},\frac{r}{s},-2)$, $E(M_5(f))=\{-1,0,1,\infty\}$, $\frac{p}{q},\frac{u}{v}\in\mathbb{Q}\backslash\{0,1,-1,\frac{1}{2},2\}$
$-1 \text{ is of type} \begin{cases} S \text{ if } \frac{p}{q}=4-\frac{r+s}{s}=n \text{ for some } n\in\mathbb{Z}; \\ T^H \text{ if } \lvert q\rvert=1 \text{ and } 4-\frac{p}{q}+\frac{1}{n}=\frac{r+s}{s} \text{ for some } n\in\mathbb{Z}; \\ T^H \text{ if } \lvert s\rvert=1 \text{ and } 4-\frac{p}{q}+\frac{1}{n}=\frac{r+s}{s} \text{ for some } n\in\mathbb{Z}; \\ Z \text{ if } \lvert s\rvert=1 \text{ or } \lvert q\rvert=1 \text{ and } 4-\frac{p}{q}+\frac{1}{n}\neq\frac{r+s}{s} \text{ for any } n\in\mathbb{Z}; \\ T \text{ otherwise.} \end{cases}$
$0 \text{ is of type} \begin{cases} S^H \text{ if } \lvert q-p\rvert=1 \text{ and } \lvert 4r+s+5q(s-r)\rvert=1; \\ T^H \text{ if } \lvert q-p\rvert=1 \text{ and } \lvert 4r+s+5q(s-r)\rvert\neq 1; \\ S^H \text{ if } \lvert s-r\rvert=1 \text{ and } \lvert 4p+q+5s(q-p)\rvert=1; \\ T^H \text{ if } \lvert s-r\rvert=1 \text{ and } \lvert 4p+q+5s(q-p)\rvert\neq 1; \\ Z \text{ otherwise.} \end{cases}$
$1 \text{ is of type} \begin{cases} T^H \text{ if } \lvert p\rvert=1 \text{ and } \lvert r\rvert=1; \\ Z \text{ if } \lvert p\rvert=1 \text{ and } \lvert r\rvert\neq 1; \\ T^H \text{ if } \lvert r\rvert=1 \text{ and } \lvert p\rvert\neq 1; \\ T \text{ otherwise.} \end{cases}$
$\infty \text{ is of type} \begin{cases} T^H \text{ if } \lvert s\rvert=1 \text{ and } \lvert q\rvert=1; \\ Z \text{ if } \lvert s\rvert=1 \text{ and } \lvert q\rvert\neq 1; \\ T^H \text{ if } \lvert q\rvert=1 \text{ and } \lvert s\rvert\neq 1; \\ T \text{ otherwise.} \end{cases}$

TABLE 4.1.13. Exceptional sets for $M_5(f)$ for f in Table 1.3.1, part 6/6.

4.2. Exceptional filling pairs at maximal distance

We are interested in enumerating all exceptional filling pairs $(M_5(f);\alpha_1,\alpha_2)$ of type $(\mathcal{C}_1,\mathcal{C}_2)$ of one-cusped hyperbolic $M_5(f)$'s with $\Delta(\alpha_1,\alpha_2)$ attaining the conjectural or known maximal value. Chapter 3 describes all $(M_5(f);\alpha_1,\alpha_2)$ of type $(\mathcal{C}_1,\mathcal{C}_2)$ when f factors through M_3 and Tables 4.1.2-4.1.13 describe all $(M_5(f);\alpha_1,\alpha_2)$ of type $(\mathcal{C}_1,\mathcal{C}_2)$ when f is equivalent to a filling instruction in Tables 1.3.1-1.3.3.

In this section we will use Corollaries 4.1.2 and 4.1.3 to consider the remaining $(M_5(f);\alpha_1,\alpha_2)$ of type $(\mathcal{C}_1,\mathcal{C}_2)$.

Δ_0	S	T	S^H	T^H
S	1	-	0 (c)	1
T		-	2	-
S^H			0	1
T^H				1

TABLE 4.2.1. Maximal distances that are ≤ 2.

Note that $\Delta(\alpha,\beta) \leq 2$ for all α, $\beta \in E(M_5(f))$ with f not factoring through M_3 and not equivalent to a filling instruction in Tables 1.3.1-1.3.3. Therefore we cannot have $\Delta(\alpha,\beta) > 2$. The pairs of type $(\mathcal{C},\mathcal{D})$ for which we can hope to realize $\Delta_0(\mathcal{C},\mathcal{D})$ using one of the $M_5(f)$'s described above are those for which the entry in Table 0.0.1 is 1 or 2. Since we will refrain from discussing the case where $\mathcal{C}$ or $\mathcal{D}$ is Z, the relevant $(\mathcal{C},\mathcal{D})$-pairs and values of $\Delta_0(\mathcal{C},\mathcal{D})$ are those in Table 4.2.1.

We start with the case where one class is reducible.

LEMMA 4.2.1. *Every hyperbolic filling instruction f not equivalent to a filling instructions in Tables 1.3.1-1.3.3 and not factoring through M_3 with $M_5(f)$ a one-cusped hyperbolic manifold with a reducible slope in $E(M_5(f))$ is equivalent to a filling instruction in Table 4.2.2.*

PROOF. We know from Tables 4.1.2-4.1.13 that if f is found in Tables 1.3.1-1.3.3 with $M_5(f)(\alpha)$ reducible for some slope α on $M_5(f)$ then $\alpha \in \{-1,0,1,\infty\}$. This means that either $E(M_5(f)) = \{0,1,\infty\}$ or that f is equivalent to a filling instruction of the form $(-1,\frac{p}{q},\frac{r}{s},\frac{u}{v})$ and $E(M_5(f)) = \{-1,0,1,\infty\}$. We now use the conditions from Lemma 4.1.1 to enumerate all such (f,α) with $M_5(f)(\alpha)$ reducible.

For $M_5(f)(0) = M_5(\frac{i_2-j_2}{i_2},\frac{j_1}{j_1-i_1},\frac{i_3+j_3}{j_3},-\frac{j_0}{i_0})(0)$ to be reducible we need one of $i_k = 0$, or $i_0 = i_2 = 1, j_0+j_2 = 0$, or $i_1 = i_3 = 1, j_1+j_3 = 0$. Now $i_k = 0$ implies that $M_5(f)$ is non-hyperbolic, $i_0 = i_2 = 1, j_0+j_2 = 0$ implies that $f = (1+n,\frac{j_1}{j_1-i_1},\frac{i_3+j_3}{j_3},-n)$, and $i_1 = i_3 = 1, j_1+j_3 = 0$ implies that $f = (\frac{i_2-j_2}{i_2},\frac{n}{n-1},\frac{n-1}{n},\frac{j_0}{i_0})$. Thus, we see that $M_5(f)(0)$ is reducible exactly when $f = (1+n,\frac{j_1}{j_1-i_1},\frac{i_3+j_3}{j_3},-n)$ or $(\frac{i_2-j_2}{i_2},\frac{n}{n-1},\frac{n-1}{n},\frac{j_0}{i_0})$.

Similarly we see that $M_5(f)(1)$ is reducible exactly when $f = (\frac{n+1}{n}, \frac{i_1}{j_1}, -\frac{1}{n}, \frac{i_3}{j_3} + 1)$ or $(\frac{i_0}{j_0}+1, \frac{1}{n}, \frac{i_2}{j_2}, \frac{n-1}{n})$, and $M_5(f)(\infty)$ is reducible exactly when $f = (n, \frac{1}{n}, \frac{j_3}{i_3}, -\frac{i_1}{j_1})$ or $(-\frac{i_2}{j_2}, \frac{j_0}{i_0}, n, \frac{1}{n})$, and $M_5(f)(-1)$ reducible exactly when $f = (-1, 1-n, \frac{i_3+2j_3}{j_3}, n)$. □

f	α	$M_5(f)(\alpha)$
$(-1, 1-n, \frac{p+2q}{q}, n)$	-1	$\mathbb{RP}^3 \# L(p, q)$
$(\frac{p-q}{p}, \frac{n}{n-1}, \frac{n-1}{n}, \frac{s}{r}), (1+n, \frac{q}{q-p}, \frac{r+s}{s}, -n)$	0	$L(p, q) \# L(r, s)$
$(\frac{n+1}{n}, \frac{p}{q}, -\frac{1}{n}, 1+\frac{r}{s}), (1+\frac{p}{q}, \frac{1}{n}, \frac{r}{s}, \frac{n-1}{n})$	1	$L(p, q) \# L(r, s)$
$(\frac{1}{n}, n, -\frac{p}{q}, \frac{s}{r}), (\frac{q}{p}, -\frac{r}{s}, n, \frac{1}{n})$	∞	$L(p, q) \# L(r, s)$

TABLE 4.2.2. All (f, α) with hyperbolic f not factoring through M_3 not found in Tables 1.3.1-1.3.3 with reducible $M_5(f)(\alpha)$.

Definition We will say that two exceptional pairs $(m_1; \alpha_1, \beta_1)$, $(m_2; \alpha_2, \beta_2)$ of type $(\mathcal{C}, \mathcal{D})$ are equivalent if there exists a homeomorphism $h : m_1 \to m_2$ with $h(\alpha_1) = \beta_2$ and $h(\alpha_1) = \beta_2$

THEOREM 4.2.2. *Every exceptional (S, T)-pair $(M_5(f); \alpha, \beta)$ with $\Delta(\alpha, \beta) = \Delta_0(S, T)$ and $M_5(f)$ a one-cusped hyperbolic manifold is equivalent to an element of*

$$\{(M_3(n, 4-n); 0, 3) : n \neq 1, 2, 3, 4\}.$$

Moreover, $E(M_3(n, 4-n)) = \{0, 1, 2, 3, \infty\}$, and

$$\begin{cases} M_3(n,4-n)(0) = \mathbb{RP}^3 \# L(3,1) \\ M_3(n,4-n)(1) = \big(S^2, (2,1), (n-3,1), (n-1,-1)\big) \\ M_3(n,4-n)(2) = \big(S^2, (3,2), (n-2,1), (n+2,2)\big), \text{ for } n \neq -1 \\ M_3(n,4-n)(2) = L(15,-11), \text{ for } n = -1 \\ M_3(n,4-n)(3) = \big(D, (2,1), (n-1,-1)\big) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \big(D, (2,1), (2n-4, n-4)\big) \\ M_3(n,4-n)(\infty) = L(n^2-4n+1, n-4) \end{cases}$$

PROOF. It is known that $\Delta_0(S, T) = 3$ and so all exceptional pairs of type (S, T) at maximal distance either factor through M_3 or are found in Tables 4.1.2-4.1.13. The (S, T)-pairs factoring through M_3 are described in Proposition 3.2.4

and no pair $(M_5(f);\alpha,\beta)$ with $\Delta_0(\alpha,\beta)=3$ in Tables 4.1.2-4.1.13 is of type (S,T). Thus every exceptional (S,T)-pair $(M_5(f);\alpha,\beta)$ with $\Delta(\alpha,\beta)=\Delta_0(S,T)$ is equivalent to an element in $\{(M_3(n,4-n);0,3) : n\neq 1,2,3,4\}$. The exceptional sets and filling descriptions in Theorem 4.2.2 are straight forward consequences of Theorems 3.0.1 and 1.1.3. This completes the proof. □

Remark The Eudave-Muñoz and Wu examples of exceptional (S,T)-pairs at maximal distance in [**EW**] are all described by pairs of the form $(M_5(f);\alpha,\beta)$. In [**Kan**] Kang shows that these examples describe every exceptional (S,T)-pair at maximal distance; thus the $(M_5(f);\alpha,\beta)$'s from Theorem 4.2.2 describe all (S,T)-pair at maximal distance. Moreover, Kang shows that the reducible filling of an exceptional (S,T)-pair at maximal distance is necessarily $\mathbb{RP}^3\#L(3,1)$. The description of the examples in Theorem 4.2.2 is consistent with the examples from [**EW**].

Questions and remarks.

Bireducible pairs and trireduciple triples: Hoffman and Matignon remark in [**HM**], that there are no known examples of bireducible pairs with no summand homeomorphic to either $L(2,1)$, $L(3,1)$, or $L(4,1)$, or trireducible triples, and ask whether such examples exists. Moreover we know from Chapter 3 and Proposition 4.1.4 that all examples coming from M_5 of exceptional bireducible pairs and trireduciple triples can be constructed using Table 4.2.2. It is not hard to see that no hyperbolic knot of the form $M_5(f)$ has two (or three) reducible fillings.

The cabling conjecture: The cabling conjecture says that there are no exceptional pairs of type (S^H,S). We know from Tables 4.1.2-4.1.13 and Proposition 3.2.4 that if a (S^H,S)-pair of the form $(M_5(f);\alpha,\beta)$ exists then f does not factor through M_3 and f is not equivalent to a filling instruction in Tables 4.1.2-4.1.13.

The first problem that we highlight is to use the conditions from Corollaries 4.1.2 and 4.1.3 to determine if any $(M_5(f);\alpha,\beta)$ with f not equivalent to a filling instruction in Tables 1.3.1-1.3.3 or factoring through M_3 is a counterexample to the cabling conjecture.

$E((m;\alpha,\beta))$ **for** $(m;\alpha,\beta)$ **of type** (C,D)**:** Classifying the sets of exceptional slopes and corresponding filling types of all manifolds realising $\Delta_0(\mathcal{C},\mathcal{D})$ is a natural extension of the Gordon program. In Theorem 4.2.2 we saw the sets of all exceptional slopes and fillings of all one boundary manifolds realising $\Delta_0(S,T)$. So, it is a natural problem to use similar arguments to those above to give a complete surgery description of all Eudave-Muñoz knots and Theorem 1.1.3 to describe the sets of exceptional slopes and corresponding fillings of the Eudave-Muñoz knots.

We remark that *every* exceptional (S^H,T)-pair at maximal distance is equivalent to one described in Proposition 3.2.3, or one described in Tables 4.1.2-4.1.13, or one of the form $(M_5(-1,\frac{p}{q},\frac{r}{s},\frac{u}{v},\frac{x}{y});\alpha,\beta)$ where $(-1,\frac{p}{q},\frac{r}{s},\frac{u}{v},\frac{x}{y})$ does not factor through M_3 and $(\alpha,\beta)=(1,-1)$ or $(-1,1)$. The second problem we highlight is to use Corollary 4.1.3 to enumerate all $(M_5(-1,\frac{p}{q},\frac{r}{s},\frac{u}{v},\frac{x}{y});\alpha,\beta)$ where $(-1,\frac{p}{q},\frac{r}{s},\frac{u}{v},\frac{x}{y})$ does not factor through M_3 and $(\alpha,\beta)=(1,-1)$ or $(-1,1)$.

4.3. Berge knots

The Berge knots are conjectured to be the only knots in S^3 admitting lens space surgeries. They are classified into 13 types. In [**Bak**], Baker describes all Berge knots of types I-VI and IX-XII by surgery instructions on 5CL. We reproduce the results of his classification in Theorem 4.3.1 as filling instructions on M_5.

Theorem 4.3.1. (Baker) *The Berge knots Types* I-VI *and* IX - XII *are completely described in Table 4.3.1 where* $n,k,p,q,r,s\in\mathbb{Z}$ *with* $\Delta(\frac{p}{q},\frac{r}{s})=1$ *and* $|\epsilon|=1$.

Remark The minimally twisted 5-chain link is unique up to homeomorphisms of S^3, and it is shown as 5CL in Figure 0.0.1, but it is not unique if we take orientation into account *i.e.* if we only look at links up to isotopy. In other words, 5CL is not isotopic to its mirror image 5CL$'$. Baker's original description does not distinguish between 5CL and 5CL$'$. The presentations in Table 4.3.1 fix the filling on M_5 (thereby reversing the sign of slopes from the original presentation for types I, IV, VI).

Remark It is easy to check that the (S^H,T^H)-pairs on the $M_5(f)$ found in Table 4.3.1 agree with Theorem 1.1.3 and satisfy the conditions of Corollaries 4.1.2 and 4.1.3.

The action of the symmetry group of M_5 gives an immediate simplification of Table 4.3.1.

Berge Type	$M_5(f)$	$(S^H, T^H)-$pair
I	$M_5(\frac{r}{s}, \frac{p}{q}, \infty, \infty)$	$(\infty, 1)$
II	$M_5(\frac{1}{2}, \frac{r+2s}{s}, \frac{p}{q}, \infty)$	$(\infty, 1)$
III	$M_5(\frac{1}{2}, \frac{4\epsilon+(4p+\epsilon)k+4\epsilon n+(4p+2\epsilon)kn}{\epsilon+pk+2\epsilon n+(2p+\epsilon)kn}, \frac{p}{2p+\epsilon}, \frac{-1}{n})$	$(\infty, 1)$
IV	$M_5(\frac{1}{n}, \frac{(2n-1)\epsilon+(2n-1)pk+(n-1)\epsilon k}{2\epsilon+2pk+\epsilon k}, -1, \frac{1+\epsilon p}{\epsilon p})$	$(\infty, 1)$
V	$M_5(\frac{1}{2}, \frac{3-2n}{1-n}, \frac{(2-n)\epsilon+(2p+\epsilon-pn)k}{\epsilon+pk}, \frac{p}{p+\epsilon})$	$(\infty, 1)$
VI	$M_5(\frac{1}{n+2}, \frac{1}{2}, \frac{-2-3n+k(6n+7)}{k+(-1+2k)(n+1)}, \frac{1}{3})$	$(\infty, 1)$
IX	$M_5(-2, \frac{4}{3}, n+1, \frac{2}{1+2n})$	$(\infty, 0)$
X	$M_5(-3, \frac{3}{2}, n+1, \frac{2}{1+2n})$	$(\infty, 0)$
XI	$M_5(\frac{3}{4}, -\frac{1}{2}, \frac{2n-1}{2}, \frac{1}{n})$	$(\infty, 1)$
XII	$M_5(\frac{2}{3}, -\frac{1}{3}, \frac{2n-1}{2}, \frac{1}{n})$	$(\infty, 1)$

TABLE 4.3.1. Surgery descriptions of Type I - VI, IX - XII Berge knots.

PROPOSITION 4.3.2. *Let* $\mathrm{IX}_n = M_5(-2, \frac{4}{3}, n+1, \frac{2}{1+2n})$, $\mathrm{X}_n = M_5(-3, \frac{3}{2}, n+1, \frac{2}{1+2n})$, $\mathrm{XI}_n = M_5(\frac{3}{4}, -\frac{1}{2}, \frac{2n-1}{2}, \frac{1}{n})\}$, $\mathrm{XII}_n = M_5(\frac{2}{3}, -\frac{1}{3}, \frac{2n-1}{2}, \frac{1}{n})$ *be the exteriors of Berge knots of types* IX, X, XI, XII. *Then*

$$\mathrm{IX}_n\big(\tfrac{x}{y}\big) \cong \mathrm{XII}_{n+1}\big(\tfrac{y-x}{y}\big) \text{ and } \mathrm{X}_n\big(\tfrac{p}{q}\big) \cong \mathrm{XI}_{n+1}\big(\tfrac{q-p}{q}\big).$$

Moreover, all $(M_5(f); \alpha, \beta)$*-pairs found in Table 4.3.1 with* f *hyperbolic are equivalent to some* $(M; \alpha', \beta')$*-pair found in Table 4.3.2.*

PROOF. The identification between types IX and XII, and X and XI is a straightforward application of Theorem 1.0.7 as is the description of types III-VI and IX-X (the type VI description also uses Theorem 2.1.1). It is clear from Theorem 1.1.3 that any Berge knot of type I or II is non-hyperbolic. Thus Table 4.3.2 contains all hyperbolic Berge knots of type I-VI, IX-XII. □

Types IX, X were known to be equivalent to types XII, XI respectively, see [**Ras**] for example, however the identification in Proposition 4.3.2 was not.

The following result is a straightforward corollary of the results in Chapter 1.

Berge Type	$\{M_5(f)\}$	$(S^H, T^H)-$pair
III	$M_5(-1, -\frac{p+\epsilon}{p}, \frac{n}{n+1}, \frac{4\epsilon+(4p+\epsilon)k+4\epsilon n+(4p+2\epsilon)kn}{\epsilon+pk+2\epsilon n+(2p+\epsilon)kn})$	$(0, \infty)$
IV	$M_5(-1, \frac{\epsilon p}{1+\epsilon p}, \frac{(3-2n)(\epsilon+pk)+(2-n)\epsilon k}{2(\epsilon+pk)+\epsilon k}, \frac{1}{1-n})$	$(\infty, 0)$
V	$M_5(-1, \frac{(1-n)\epsilon+k(\epsilon+p-np)}{\epsilon+pk}, \frac{p+\epsilon}{\epsilon}, \frac{3-2n}{1-n}$	$(0, \infty)$
VI	$M_3(\frac{-2-3n+(6n+7)k}{-1-n+k+2kn}, \frac{3n+4}{n+1})$	$(1, \infty)$
IX	$M_5(-2, \frac{4}{3}, n+1, \frac{2}{1+2n})$	$(\infty, 0)$
X	$M_5(-2, \frac{4}{3}, \frac{1-2n}{2}, -\frac{1}{n})$	$(\infty, 0)$

Table 4.3.2. Filling descriptions of hyperbolic Berge knots.

Corollary 4.3.3. *Let* $\mathrm{IX}_n = M_5(-2, \frac{4}{3}, n+1, \frac{2}{1+2n})$ *and* $\mathrm{X}_n = M_5(-2, \frac{4}{3}, \frac{1-2n}{2}, -\frac{1}{n})$ *be the exteriors of Berge knots of type* IX *and* X *respectively. Then* IX_n *and* X_n *are hyperbolic except when* $n \in \{-1, 0\}$ *and we have*

$$E(\mathrm{IX}_n) = \begin{cases} \{0,1,\infty\}, & (T^H, T, S^H)\textit{-triple if } n \notin \{-2, 1\} \\ \{0,1,2,\infty\}, & (T^H, Z, T, S^H)\textit{-quadruple if } n = -2 \\ \{-1, -\frac{1}{2}, 0, 1, \infty\}, & (T^H, T, T^H, Z, S^H)\textit{-quintuple if } n = 1, \end{cases} \tag{4.1}$$

$$E(\mathrm{X}_n) = \begin{cases} \{0,1,\infty\}, & (T^H, T, S^H)\textit{-triple if } n \notin \{-2, 1\} \\ \{0,1,2,\infty\}, & (T^H, Z, T, S^H)\textit{-quadruple if } n = -2 \\ \{-1, 0, 1, 2, \infty\}, & (T, T^H, Z, T, S^H)\textit{-quintuple if } n = 1. \end{cases} \tag{4.2}$$

Questions and remarks. We end our discussion of the Berge knots with the following problems:

Sets of exceptional fillings of the Berge knots: Using the results from Chapter 1 and Table 4.3.2 it is clearly possible to extend Corollary 4.3.3 to describe the sets of exceptional slopes and the corresponding fillings of all hyperbolic Berge knots not of type VII or VIII.

Knots with cyclic surgeries: We have seen that all knot exteriors in S^3 with a lens space filling of the form $M_5(f)$ are described in Proposition 3.2.3, or have f in Tables 4.1.2-4.1.13, or can be completely described, after a lengthy analysis, using Corollaries 4.1.2-4.1.3.

Denote the set of all knot exteriors in of the form $M_5(f)$ with a lens space filling by $\mathcal{B}$. The knot exteriors in Table 4.3.2 are easily seen to belong to $\mathcal{B}$. A problem worth facing is then: is Table 4.3.2 exactly equal to $\mathcal{B}$, and, if not, are the examples from $\mathcal{B}$ not found in Table 4.3.2 Berge knots or counterexamples to the Berge conjecture?

4.4. Intersection index

We now recall that $\mathcal{I}$ denotes the class of toroidal manifolds with intersection index greater than 1, as defined in Section 3.3. We will now describe $E(M_5(f))$ for all f such that $M_5(f)$ is a hyperbolic manifold with an exceptional slope of type-$\mathcal{I}$. There are precisely three such $M_5(f)$'s factoring through M_3, as described in Table 3.3.1. To describe the other relevant f's, *i.e.* those giving a hyperbolic $M_5(f)$ not factoring through M_3 with some slope α of type $\mathcal{I}$, we note that if $M_5(f)(\alpha)$ is of type-$\mathcal{I}$ then (f, α) is one of the following filling instructions:

$$\begin{aligned}\{&(-2,-2,-2,-2,-2),(-1,-3,-2,-2,-3),(3,-\tfrac{1}{2},-2,-\tfrac{1}{2},3),\\ &\quad(-1,-2,-3,-2,-4),(-3,-1,-4,-1,-3),(-1,-3,-1,-4,-4),\\ &\qquad(-1,-2,-4,-1,-5),(-2,-2,-1,-5,-3),(-3,-1,-2,-4,-3)\}.\end{aligned}$$

It is now easy to enumerate all (f, α)'s with f not factoring through M_3; for example, the filling instruction $(-1,-3,-2,-2,-3)$ corresponds to (f, α) being one of $((-1,-3,-2,-2),-3)$ or $((-3,-2,-2,-3),-1)$ or $((-2,-3,-1,-3),-2)$. We use Theorem 1.0.7 to reduce the number of (f, α)'s and also remark that many of the $M_5(f)$'s with $f \sim (-1, \frac{p}{q}, \frac{r}{s}, \frac{u}{v})$ can seen to be homeomorphic using Theorem 2.1.1. The resulting list of (f, α)'s corresponding to non-homeomorphic $M_5(f)$'s with f not factoring through M_3 and $M_5(f)$ admitting a type-$\mathcal{I}$ filling are described in Tables 4.4.1-4.4.4 along with $E(M_5(f)$ and the corresponding fillings.

$M_5(-1,-3,-2,-2)(\alpha)$	
$\alpha=-3:$ $\big(D,(2,1),(2,-1)\big)\bigcup_{\begin{pmatrix}1&4\\1&3\end{pmatrix}}\big(D,(2,-3),(3,2)\big)$	$\alpha=\infty:\quad L(4,3)$
$\alpha=1:$ $\big(D,(2,-1),(2,-1)\big)\bigcup_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}\big(D,(3,-1),(3,-1)\big)$	$\alpha=0:\quad \big(S^2,(4,3),(3,-1),(4,-1)\big)$
$\alpha=-1:\ \big(S^2,(6,-1),(4,-1),(2,1)\big)$	$\alpha=-2:\quad \big(S^2,(2,-1),(3,1),(7,1)\big)$
$M_5(-3,-2,-2,-3)(\alpha)=M_5(-2,\frac{1}{4},\frac{3}{2},\frac{4}{3})(\alpha')$	
$\alpha=-1:$ $\big(D,(2,1),(2,-1)\big)\bigcup_{\begin{pmatrix}1&4\\1&3\end{pmatrix}}\big(D,(2,-3),(3,2)\big)$	$\alpha=\infty:\quad L(16,7)$
$\alpha=1:$ $\big(D,(2,-1),(4,-1)\big)\bigcup_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}\big(D,(2,-1),(4,-1)\big)$	$\alpha=0:\quad \big(S^2,(7,-1),(3,-1),(3,2)\big)$

Table 4.4.1. Sets of exceptional slopes and fillings of hyperbolic manifolds with a filling equal to $M_5(-1,-3,-2,-2,-3)\in\mathcal{I}$.

$M_5(-\frac{1}{2},3,3,-\frac{1}{2})(\alpha') = M_5(-2,\frac{3}{2},\frac{3}{2},-2)(\alpha)$ $\alpha=\infty:$ $\big(D,(2,-3),(2,-1)\big)\bigcup_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}\big(D,(2,-3),(2,-1)\big)$ $\alpha=-\frac{1}{2}:$ $\big(A,(2,-1)\big)/\begin{pmatrix}1&2\\1&1\end{pmatrix}$ $\alpha=-1:$ $\big(D,(2,9),(2,11)\big)\bigcup_{\begin{pmatrix}0&1\\-1&1\end{pmatrix}}\big(D,(2,1),(3,1)\big)$ $\alpha=0:$ $L(24,5)$ $\alpha=1:$ $\big(D,(3,-1),(3,2)\big)\bigcup_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}\big(D,(3,-1),(3,2)\big)$
$M_5(-2,-\frac{1}{2},3,3)(\alpha)$ $\alpha=-\frac{1}{2}:\big(A,(2,-1)\big)/\begin{pmatrix}1&2\\1&1\end{pmatrix}$ $\alpha=\infty:\big(S^2,(2,1),(2,-1),(8,3)\big)$ $\alpha=-1:\big(S^2,(2,1),(4,-1),(5,-2)\big)$ $\alpha=0:L(3,1)\#L(2,1)$ $\alpha=1:\big(S^2,(3,1),(3,-1),(3,2)\big)$
$M_5(-2,-3,-2,-4)(\alpha)$ $\alpha=-1:$ $\big(D,(2,1),(2,1)\big)\bigcup_{\begin{pmatrix}2&3\\-1&-2\end{pmatrix}}\big(D,(2,1),(3,1)\big)$ $\alpha=\infty:$ $L(27,17)$ $\alpha=1:$ $\big(D,(3,-1),(2,-1)\big)\bigcup_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}\big(D,(3,-1),(5,-1)\big)$ $\alpha=0:$ $\big(S^2,(7,-1),(4,3),(3,-1)\big)$
$M_5(-1,-4,-2,-3)(\alpha)$ $\alpha=-2:$ $\big(D,(2,1),(2,1)\big)\bigcup_{\begin{pmatrix}2&3\\-1&-2\end{pmatrix}}\big(D,(2,1),(3,1)\big)$ $\alpha=\infty:$ $L(50,9)$ $\alpha=1:$ $\big(D,(2,-1),(2,-1)\big)\bigcup_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}\big(D,(4,-1),(4,-1)\big)$ $\alpha=0:$ $\big(S^2,(5,4),(3,-1),(5,-1)\big)$ $\alpha=-1:\big(S^2,(4,-1),(2,1),(8,-1)\big)$

TABLE 4.4.2. Sets of exceptional slopes and fillings of hyperbolic manifolds with a filling equal to $M_5(-2,-\frac{1}{2},3,3,-\frac{1}{2}) \in \mathcal{I}$ or $M_3(-2,-2,-2)$.

$M_5(-1,-3,-3,-4)(\alpha)$	
$\alpha=-2:$ $\big(D,(2,1),(2,1)\big)\bigcup_{\begin{pmatrix}1&2\\0&-1\end{pmatrix}}\big(D,(2,1),(3,1)\big)$	$\alpha=\infty:$ $L(18,11)$
$\alpha=-1:$ $\big(S^2,(8,-1),(2,1),(5,-1)\big)$	$\alpha=0:$ $\big(S^2,(6,-1),(4,3),(4,-1)\big)$
$\alpha=1:$ $\big(D,(2,-1),(3,-1)\big)\bigcup_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}\big(D,(3,-1),(5,-1)\big)$	
$M_5(-2,-2,-3,-5)(\alpha)$	
$\alpha=-1:$ $\big(D,(2,1),(2,1)\big)\bigcup_{\begin{pmatrix}1&2\\0&-1\end{pmatrix}}\big(D,(2,1),(3,1)\big)$	$\alpha=\infty:$ $L(32,23)$
$\alpha=1:$ $\big(D,(3,-1),(3,-1)\big)\bigcup_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}\big(D,(2,-1),(6,-1)\big)$	$\alpha=0:$ $\big(S^2,(6,-1),(2,-1),(3,2)\big)$
$M_5(-2,-4,-3,-3)(\alpha)$	
$\alpha=-1:$ $\big(D,(2,1),(2,1)\big)\bigcup_{\begin{pmatrix}1&2\\0&-1\end{pmatrix}}\big(D,(2,1),(3,1)\big)$	$\alpha=\infty:$ $L(50,29)$
$\alpha=1:$ $\big(D,(3,-1),(3,-1)\big)\bigcup_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}\big(D,(4,-1),(4,-1)\big)$	$\alpha=0:$ $\big(S^2,(6,-1),(5,4),(4,-1)\big)$

TABLE 4.4.3. Sets of exceptional slopes and fillings of hyperbolic manifolds with a filling equal to $M_3(-1,-3,-3)=M_5(g)\in\mathcal{I}$.

$M_5(g)=M_5(-2,-2,-2,-2)$	
$-2:$ $\big(D,(2,1),(2,-1)\big)\bigcup_{\begin{pmatrix}1&4\\1&3\end{pmatrix}}\big(D,(2,-3),(3,2)\big)$	$\infty:$ $L(5,4)$
$-1:$ $\big(S^2,(2,-1),(3,1),(7,1)\big)$	$0:$ $\big(S^2,(5,-1),(3,-1),(3,2)\big)$
$1:$ $\big(D,(2,1),(3,1)\big)\bigcup_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}\big(D,(2,1),(3,1)\big)$	

TABLE 4.4.4. $E(M_5(-2,-2,-2,-2))$.

Questions. In light of Tables 4.4.1- 4.4.4 we end this thesis with the following questions and problems;

Exceptional $\mathcal{I}$-fillings examples: We are unaware of examples of exceptional $\mathcal{I}$-fillings not presented in this thesis. The result of [**Mye**] imply that there exist infinitely many hyperbolic N's with some type-$\mathcal{I}$ filling $N(\alpha)$ but we would like to make the construction explicit and exhibit an infinite family of such (N,α)'s.

Theorem 1.1.1 shows that $M_n(-2,\cdots-2)$ is of type-$\mathcal{I}$ for $1<n<6$. If we define nCL to be the n-chain link with the k^{th} and $(k+1)^{\text{th}} \mod n$ components linked with linking number 1 so that $n\text{CL}(-1)=(n-1)\text{CL}$ we can ask if $M_n(-2,\cdots-2)$ is of type $\mathcal{I}$ for every n? Moreover, for $1<n<6$, each $M_n(-2,\ldots,2)$ is a graph manifold with intersection number n. Is this a coincidence or true for all n?

The Gordon program for $\mathcal{I}$: On the basis of the 13 examples of Tables 4.4.1- 4.4.4 and 3.3.1 we have the following lower bounds on the maximal distance $\Delta_0(\mathcal{I},X)$ for $X\in\{S^H,S,T^H,T,\mathcal{I},Z\}$:

	S^H	S	T^H	T	$\mathcal{I}$	Z
$\mathcal{I}$	$-\infty$	1	1	4	0	5

TABLE 4.4.5. Lower bounds on $\Delta_0(\mathcal{I},\mathcal{C})$.

The lower bounds in Table 4.4.5 are obtained from a small sample set, so we ask: what is $\Delta_0(\mathcal{I},\mathcal{C})$ for $\mathcal{C}\in\{S^H,S,T^H,T,\mathcal{I},Z\}$?

Lens space knot complements: All the hyperbolic manifolds from Tables 4.4.1-4.4.4 whose type-$\mathcal{I}$ filling consists of a graph manifold with two vertices are Hyperbolic knot complement in a lens space. Does a unique lens space filling always exist when a manifold contains an exceptional type-$\mathcal{I}$ filling with two vertices? If so, are the type-$\mathcal{I}$-fillings always realised by integral fillings?

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